MAE 104 - HOMEWORK # 2:

PROBLEM 1:

1)

* The uniform flow and two vortices present the following streamfunctions:

- $\psi_u(x,y) = U_\infty \cdot y$
- $\psi_{V+}(x,y) = \frac{\Gamma}{2\pi} \ln\left(\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2}\right)$
- $\psi_{V-}(x,y) = -\frac{\Gamma}{2\pi} \ln\left(\sqrt{\left(\frac{x}{a}-1\right)^2 + \left(\frac{y}{a}\right)^2}\right)$

* The streamfunction of the total flow is, then:

- $$\boxed{\psi(x,y) = U_\infty \cdot y + \frac{\Gamma}{2\pi} \cdot \left[\ln\left(\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2}\right) - \ln\left(\sqrt{\left(\frac{x}{a}-1\right)^2 + \left(\frac{y}{a}\right)^2}\right)\right]}$$

2)

* $u = \partial\psi/\partial y$

* $v = -\partial\psi/\partial x$

* $$\vec{\omega} = \nabla \wedge \vec{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\nabla^2\psi \end{Bmatrix} = \bar{0} \Rightarrow$$

$$\Rightarrow \boxed{\vec{\omega} = \nabla \wedge \vec{V} = \vec{0}}$$

* Since $\vec{\omega} = \vec{0}$, then $\phi(x,y)$ does exist.

* $$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = U_\infty + \frac{\Gamma}{2\pi} y \left(\frac{1}{x^2+y^2} - \frac{1}{(x-a)^2+y^2}\right) \Rightarrow$$

$$\Rightarrow \phi_u(x,y) = \int \frac{\partial\psi}{\partial y} dx + f(y) = U_\infty \cdot x - \frac{\Gamma}{2\pi}\left[\arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{y}{x-a}\right)\right].$$

$$* \; v = \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} = -\frac{\Gamma}{2\pi}\left(\frac{x}{x^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right) \Rightarrow$$

$$\Rightarrow \phi_v(x,y) = \int -\frac{\partial \psi}{\partial x}\,dy + g(x) = \frac{-\Gamma}{2\pi}\left[\arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{y}{x-a}\right)\right] + g(x)$$

$$* \; \phi_u(x,y) = \phi_v(x,y) \Rightarrow U_\infty \cdot x + f(y) = g(x) \Rightarrow \begin{cases} f(y) = 0 \\ g(x) = U_\infty \cdot x \end{cases}$$

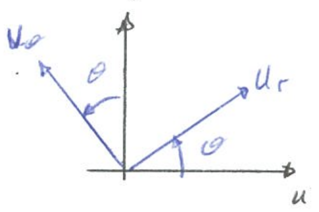
$$* \; \boxed{\phi(x,y) = U_\infty \cdot x - \frac{\Gamma}{2\pi}\left[\arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{y}{x-a}\right)\right]}$$

3)

$$* \; \boxed{u(x,y) = \frac{\partial \psi}{\partial y} = U_\infty + \frac{\Gamma}{2\pi}\, y\left(\frac{1}{x^2+y^2} - \frac{1}{(x-a)^2+y^2}\right)}$$

$$* \; \boxed{v(x,y) = -\frac{\partial \psi}{\partial x} = -\frac{\Gamma}{2\pi}\left(\frac{x}{x^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right)}$$

* To calculate the velocity in polar coordinates, we make the following change of variables:



- $x = r \cdot \cos\theta$
- $y = r \cdot \sin\theta$
- $V_r = u \cdot \cos\theta + v \cdot \sin\theta$
- $V_\theta = -u \cdot \sin\theta + v \cdot \cos\theta$

and we get:

- $\boxed{V_r(r,\theta) = U_\infty \cdot \cos\theta - \frac{\Gamma}{2\pi} \cdot \frac{a \cdot \sin\theta}{r^2+a^2-2ar \cdot \cos\theta}}$
- $\boxed{V_\theta(r,\theta) = -U_\infty \cdot \sin\theta + \frac{\Gamma}{2\pi}\,\frac{a}{r}\,\frac{r \cdot \cos\theta - a}{r^2+a^2-2ar \cdot \cos\theta}}$

* For downstream, $r \to \infty$:

- $$\boxed{\begin{array}{l} \bullet\ V_r(r\to\infty) \to U_\infty \cos\theta \\ \bullet\ V_\theta(r\to\infty) \to -U_\infty \sin\theta \end{array}} \Bigg\} \Rightarrow \boxed{\vec{V}(r\to\infty) = (u,v) \to (U_\infty, 0)}$$

4)

* $$V(x,y) = 0 \Rightarrow \frac{x}{x^2+y^2} = \frac{x-a}{(x-a)^2+y^2} \Rightarrow \frac{y}{a} = \pm\sqrt{\frac{x}{a}\left(\frac{x}{a}-1\right)}$$

* We plug these values of y in the equation for $u(x,y)$ and solve it for x:

- First use $\frac{y_+}{a} = \sqrt{\frac{x}{a}\left(\frac{x}{a}-1\right)}$:

* $$u(x, y=y_+) = 0 \Rightarrow \frac{x}{a} = \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\sqrt{1 \pm \sqrt{1+G^2}}\right]$$

where:

- $$G = \frac{2\Gamma}{\pi a U_\infty}$$

* Since $G > 0$, the "$-$" singn is not a physical solution.

* Using this value for x we get:

- $$\frac{y_+}{a} = \frac{1}{2\sqrt{2}}\sqrt{\sqrt{1+G^2}-1}$$

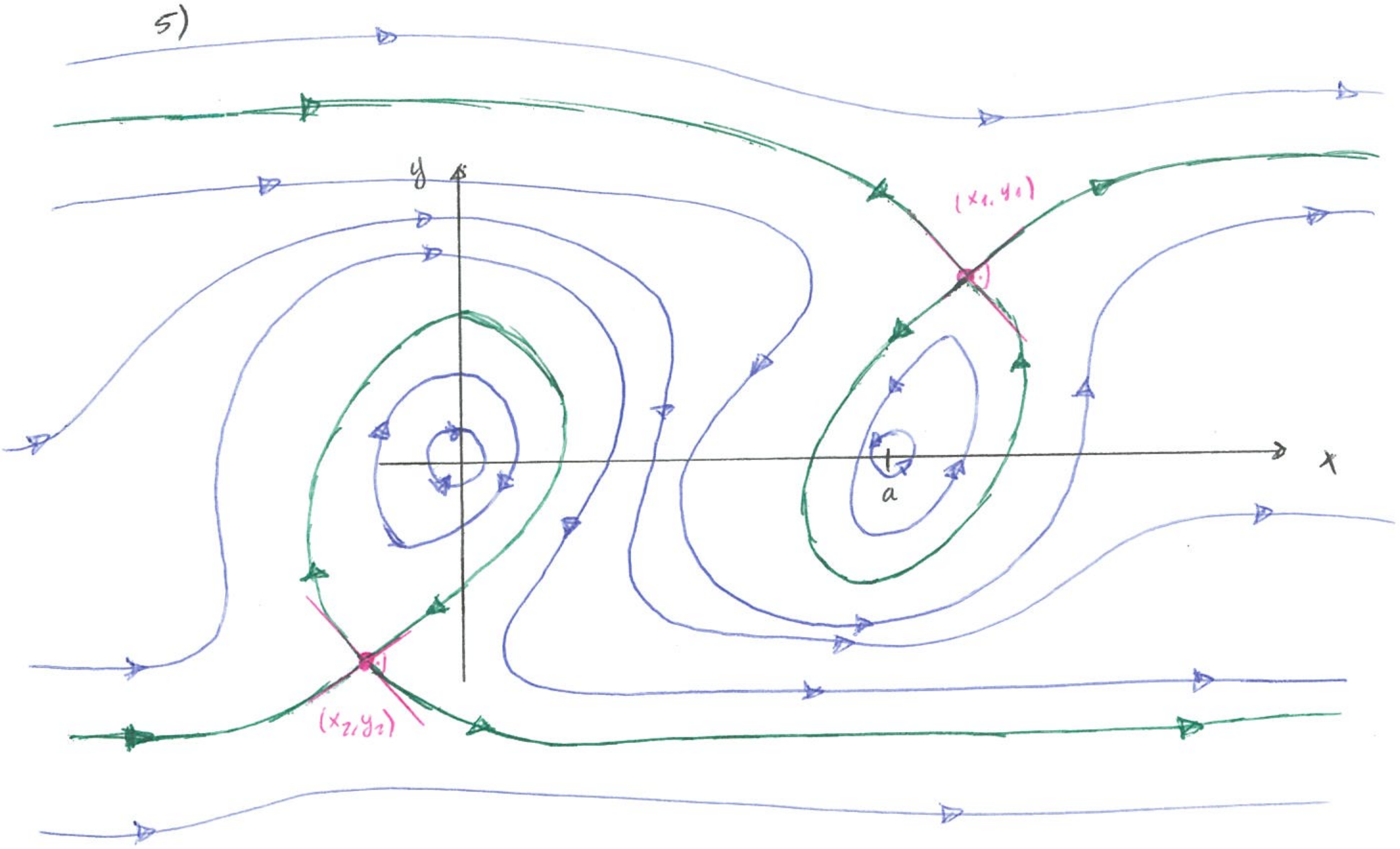
* The first stagnation point is, thus:

- $$\boxed{(x_1, y_1) = \left(\frac{a}{2}\left[1 + \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+G^2}+1}\right], \frac{a}{2\sqrt{2}}\sqrt{\sqrt{1+G^2}-1}\right)}$$

- Proceeding similarly for y_-:

* $$\boxed{(x_2, y_2) = \left(\frac{a}{2}\left[1 - \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+G^2}+1}\right], \frac{-a}{2\sqrt{2}}\sqrt{\sqrt{1+G^2}-1}\right)}$$

5)

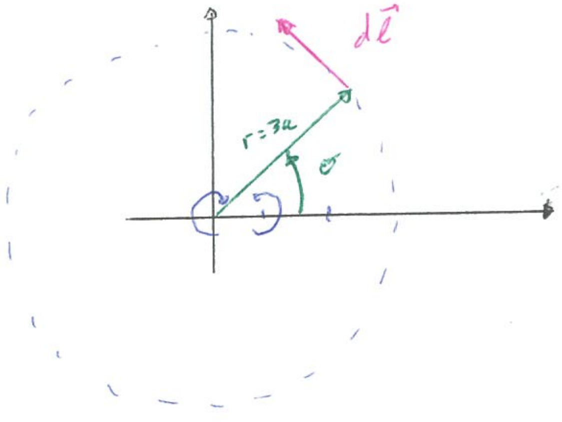


- *Stagnation points are drawn in RED.
- *Dividing streamlines are plotted in GREEN.
- *Regular streamlines are plotted in BLUE.
- *Each streamline has arrow heads pointing in the direction of the flow.

6) * The total lift is calculated by applying the Kutta-Jukowski theorem:

- $L' = \rho \cdot v_\infty \cdot \sum_i \Gamma_i = \rho \cdot v_\infty \cdot (\Gamma - \Gamma) = 0 \quad \boxed{L' = 0}$

7) * The circle of radius $r = 3a$ and centered at the origin encloses both vortices.



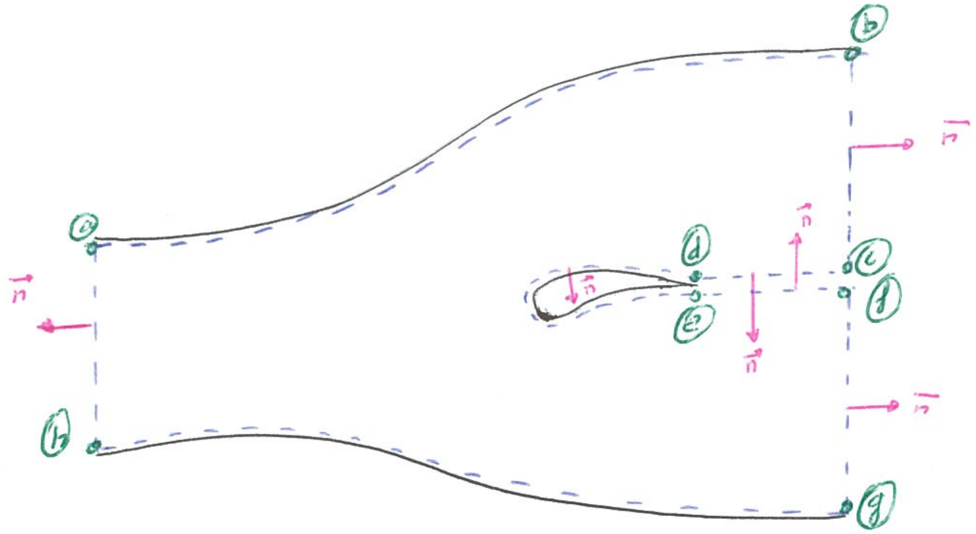
- $r = 3a$
- $dl = r\, d\theta = 3a\, d\theta$
- $\vec{V} \cdot d\vec{l} = V_\theta \cdot dl = V_\theta(r=3a, \theta) \cdot 3a \cdot d\theta$
- $$Circ = -\oint_{C_1} \vec{V} \cdot d\vec{l} = -3a \cdot \int_{\theta=0}^{\theta=2\pi} \left\{ -V_\infty \sin\theta + \frac{1}{6}\frac{\Gamma}{\pi} \frac{3\cos\theta - 1}{a \cdot [10 - 6\cos\theta]} \right\} d\theta = 0$$
- Any path enclosing both singularities has a circulation $\boxed{Circ = 0}$, as expected.

* The circle of radius $r = a/2$ encloses only the vortex at the origin:

- $r = a/2$
- $dl = \frac{a}{2} d\theta$
- $\vec{V} \cdot d\vec{l} = V_\theta(r = \frac{a}{2}, \theta) \cdot \frac{a}{2} d\theta$
- $Circ = -\oint_{C_1} \vec{V} \cdot d\vec{l} = \Gamma$
- Any path enclosing the singularity at the origin presents a circulation $\boxed{Circ = \Gamma}$, as expected.

PROBLEM 2:

1)
* I use the control volume depicted in the following sketch:



- The segments ⓐ-ⓑ, ⓓ-ⓔ and ⓖ-ⓗ are streamlines. That way, in those segments $\vec{V} \cdot \vec{n} = 0$.

- The segments ⓒ-ⓓ and ⓔ-ⓕ are infinitely closed together. Any flux comming in or out of ⓒ-ⓓ is equal and opposite to the flux comming in and out of ⓔ-ⓕ. The pressure on both segments is equal, and the normal are parallel and opposite.

- The segments ⓗ-ⓐ, ⓑ-ⓒ and ⓕ-ⓖ present known velocity and pressure.

- The pressure in the segments ⓐ-ⓑ and ⓖ-ⓗ is known.

- The integral of the pressure stresses on the segment ⓓ-ⓔ is the aerodynamic force.

2)

* Continuity equation:

$$\cdot \cancelto{0}{\frac{d}{dt}\int_{\mathcal{V}_c} \rho \, d\mathcal{V}} + \oint_{S_c} \rho \vec{v}\cdot\vec{n} \, dS = 0$$

STEADY

* Separate the integral into the different segments:

$$\cdot \int_h^a \rho\vec{v}\cdot\vec{n}\,ds + \int_a^b \cancelto{0}{\rho\vec{v}\cdot\vec{n}}\,ds + \int_b^c \rho\vec{v}\cdot\vec{n}\,ds + \int_c^d \rho\vec{v}\cdot\vec{n}\,ds + \int_d^e \cancelto{0}{\rho\vec{v}\cdot\vec{n}}\,ds + \int_e^f \rho\vec{v}\cdot\vec{n}\,ds +$$

STREAMLINE STREAMLINE

EQUAL AND OPPOSITE SIGN

$$+ \int_f^g \rho\vec{v}\cdot\vec{n}\,ds + \int_g^h \cancelto{0}{\rho\vec{v}\cdot\vec{n}}\,ds = 0$$

STREAMLINE

* The remaining integrals can be calculated as:

$$\cdot \int_h^a \rho\vec{v}\cdot\vec{n}\,ds = \rho \cdot \int_{y=-h_1/2}^{y=h_1/2} U_\infty \cdot \bar{i} \cdot (-\bar{i})\,dy = -\rho U_\infty \cdot h_1$$

$$\cdot \int_b^c \rho\vec{v}\cdot\vec{n}\,ds + \int_f^g \rho\vec{v}\cdot\vec{n}\,ds = 2\cdot\rho \int_{y=h_2/2}^{y=0} u_e \cdot \bar{i} \cdot \bar{i}\,(-dy) = 2\rho \int_{y=0}^{y=h_2/2} u_e(y)\,dy =$$

$$= 2\cdot\rho \left\{ U_\infty \cdot \int_{y=0}^{y=h_2/2} dy - A \cdot \int_{y=0}^{y=b/2} dy - A \cdot \int_{y=0}^{y=b/2} \cos\left(\frac{2\pi y}{b}\right) dy \right\} =$$

$$= 2\cdot\rho\cdot \left\{ U_\infty \frac{h_2}{2} - A \cdot \frac{b}{2} - \frac{1}{2} \frac{A b \sin(2\pi y/b)}{\pi} \Bigg|_{y=0}^{y=b/2} \right\} =$$

$$= \rho \cdot \left\{ U_\infty h_2 - Ab + 0 \right\} = \rho U_\infty h_2 - \rho A b$$

* Making their sum zero:

- $$\int_h^a \rho \vec{V}\cdot\vec{n}\, ds + \int_b^c \rho \vec{V}\cdot\vec{n}\, ds + \int_f^g \rho \vec{V}\cdot\vec{n}\, ds = 0$$

- $$-\rho U_\infty h_1 + \rho U_\infty h_2 - \rho A b = 0 \Rightarrow \boxed{b = \frac{U_\infty}{A}(h_2 - h_1)}$$

3) * Momentum equation:

- $$D' = \left\{\int_d^e p\vec{n}\, ds\right\}\cdot\vec{i} \Rightarrow$$ There is no minus sign because of the choice of the normal!!!

- $$L' = \left\{\int_d^e p\vec{n}\, ds\right\}\cdot\vec{j}$$

- $$\underset{\text{STEADY}}{\cancelto{0}{\frac{\partial}{\partial t}\int_{V_c} \rho\vec{V}\, d\mathcal{V}}} + \oint_{S_c} \rho\vec{V}(\vec{V}\cdot\vec{n})\, dS = \underset{\text{NEGLECT}}{\cancelto{0}{\int_{V_c} \rho\vec{g}\, d\mathcal{V}}} - \oint_{S_c} p\vec{n}\, dS + \underset{\text{INVISCID}}{\cancelto{0}{\oint_{S_c} \vec{n}\,\bar{\bar{\tau}}\, dS}}$$

* Separate both integrals into segments and simplify them:

- $$\oint_{S_c} \rho\vec{V}(\vec{V}\cdot\vec{n})\, dS = \int_h^a \rho\vec{V}(\vec{V}\cdot\vec{n})\, dS + \int_a^b \rho\vec{V}\underset{\text{STREAMLINE}}{\cancelto{0}{(\vec{V}\cdot\vec{n})}}\, ds + \int_b^c \rho\vec{V}(\vec{V}\cdot\vec{n})\, ds + \cancel{\int_c^d \rho\vec{V}(\vec{V}\cdot\vec{n})\, ds} +$$

EQUAL AND OPPOSITE SIGN

$$+ \int_d^e \rho\vec{V}\underset{\text{STREAMLINE}}{\cancelto{0}{(\vec{V}\cdot\vec{n})}}\, ds + \cancel{\int_e^f \rho\vec{V}(\vec{V}\cdot\vec{n})\, ds} + \int_f^g \rho\vec{V}(\vec{V}\cdot\vec{n})\, ds + \int_g^h \rho\vec{V}\underset{\text{STREAMLINE}}{\cancelto{0}{(\vec{V}\cdot\vec{n})}}\, ds =$$

$$= \rho\cdot\int_{y=-h_1/2}^{y=h_1/2} U_\infty\vec{i}\cdot\left[U_\infty\vec{i}\cdot(-\vec{i})\right] dy + \rho\int_{y=h_2/2}^{y=-h_2/2} u_e\vec{i}\cdot\left[u_e\vec{i}\cdot\vec{i}\right](-dy) =$$

$$= \rho\cdot\vec{i}\cdot\left\{-\int_{y=-\frac{h_1}{2}}^{y=h_1/2} U_\infty^2\, dy + \int_{y=-h_2/2}^{y=h_2/2} u_e^2(y)\, dy\right\} =$$

$$= \rho\vec{i}\left\{-U_\infty^2 h_2 + 2\cdot\int_{y=0}^{y=b/2}\left[U_\infty - A\left(1+\cos\left(\frac{2\pi y}{b}\right)\right)\right]^2 dy + 2\int_{y=b/2}^{y=h_2/2} U_\infty^2\, dy\right\} =$$

$$= \rho\,\vec{i}\cdot\left\{ -U_\infty^2 h_2 + \frac{3}{2}A^2 b - 2AU_\infty b + \cancel{U_\infty^2 b} + U_\infty^2 (h_2 - \cancel{b}) \right\}$$

- Using the value of b from the previous part:

$$* \oint_{S_c} \rho\vec{v}(\vec{v}\cdot\vec{n})\,dS = \rho\,\vec{i}\cdot\left\{ U_\infty^2 (h_2 - h_1) + A\cdot b\cdot\left(\frac{3}{2}A - 2U_\infty\right)\right\} =$$

$$= \rho\,\vec{i}\cdot U_\infty \cdot (h_2 - h_1)\cdot\left(\frac{3}{2}A - U_\infty\right)$$

$$\bullet \oint_{S_c} p\vec{n}\,dS = \cancelto{0}{\int_h^a p_\infty \vec{n}\,ds + \int_b^c p_\infty \vec{n}\,ds + \int_f^g p_\infty \vec{n}\,ds + \int_g^h p_\infty \vec{n}\,ds} +$$

Integral of $p_\infty \vec{n}$ along a closed surface

$$+ \cancelto{0}{\int_c^d p\vec{n}\,ds + \int_e^f p\vec{n}\,ds} + \int_d^e p\vec{n}\,ds = D'\vec{i} + L'\vec{j}$$

Equal and opposite sign.

* Equating both integrals:

$$\bullet\ D' = \oint_{S_c} p\cdot\vec{n}\,dS\cdot\vec{i} = -\oint_{S_c} \rho\vec{v}(\vec{v}\cdot\vec{n})\,dS\cdot\vec{i}$$

$$\boxed{D' = \rho\, U_\infty \left(U_\infty - \frac{3}{2}A\right)\cdot(h_2 - h_1)}$$