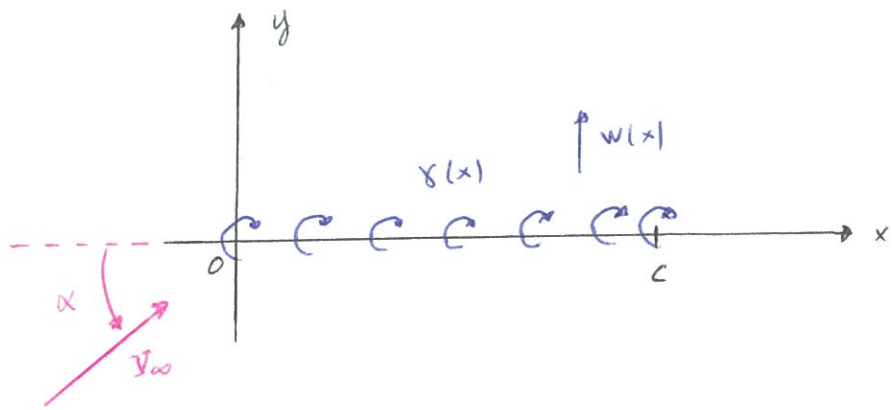
# MAE 104 - HOMEWORK #3:

## Problem 1:

1) We apply the Thin Airfoil Theory. Substitute the flat plate by a vortex sheet of intensity $\gamma(x)$:




From theory: $\boxed{\gamma(\theta) = 2 V_\infty \alpha \frac{1+\cos\theta}{\sin\theta}}$

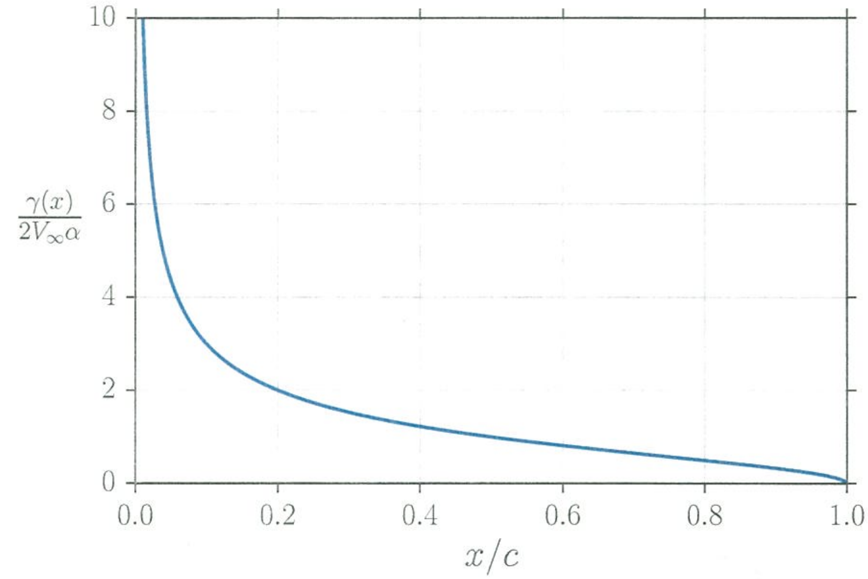
We make the change of variable:

* $x = \frac{c}{2}(1-\cos\theta)$
* $\cos\theta = 1 - 2\frac{x}{c}$
* $\sin\theta = 2 \cdot \sqrt{\frac{x}{c}\left(1-\frac{x}{c}\right)}$

and obtain $\gamma(x) = 2 V_\infty \alpha \sqrt{\frac{c}{x} - 1}$.

We can see that there is a singularity at $\gamma(x=0)$ and that $\gamma(x=c)=0$.

The sheet intensity is plotted in the following figure.




2)

From Thin Airfoil Theory, for this configuration:

$$* \; w(x) = -\int_{\xi=0}^{\xi=c} \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

We make the change of variable:

$$* \; \xi = \frac{c}{2}(1-\cos\theta) \Rightarrow \begin{cases} d\xi = \frac{c}{2}\sin\theta\, d\theta \\ \xi = 0 \Rightarrow \theta = 0 \\ \xi = c \Rightarrow \theta = \pi \end{cases}$$

We get:

$$* \; w(x) = \frac{-1}{2\pi}\int_{\theta=0}^{\theta=\pi} \frac{\gamma(\theta)\frac{c}{2}\sin\theta\, d\theta}{x - \frac{c}{2}(1-\cos\theta)} = \frac{-1}{\cancel{2}\pi}\int_{\theta=0}^{\theta=\pi} \frac{\cancel{2}V_\infty\alpha\frac{1+\cos\theta}{\sin\theta}\frac{c}{2}\sin\theta\, d\theta}{x-\frac{c}{2}(1-\cos\theta)} =$$

$$= \frac{-V_\infty\alpha}{\pi}\int_{\theta=0}^{\theta=\pi} \frac{1+\cos\theta}{\cos\theta + \left(\frac{2x}{c}-1\right)}\, d\theta =$$

$$= -\frac{V_\infty\alpha}{\pi}\left\{ \cancelto{\pi}{\int_{\theta=0}^{\theta=\pi} d\theta} + 2\left(1-\frac{x}{c}\right)\cdot\int_{\theta=0}^{\theta=\pi}\frac{d\theta}{\cos\theta+\left(\frac{2x}{c}-1\right)} \right\}$$

There are three distinct flow regions:

a) For $x < 0$ (upstream):

- $A \equiv \frac{2x}{c} - 1 < -1$

- $\displaystyle\int_{\theta=0}^{\theta=\pi} \frac{d\theta}{\cos\theta + A} = \frac{-\pi}{\sqrt{A^2-1}} = \frac{-\pi}{2\sqrt{\frac{x}{c}\left(\frac{x}{c}-1\right)}}$

- $\displaystyle w(x)\Big|_{x<0} = \frac{-V_\infty \alpha}{\pi}\left[\pi + \cancel{2}\left(1-\frac{x}{c}\right)\cdot\frac{-\pi}{\cancel{2}\sqrt{\frac{x}{c}\left(\frac{x}{c}-1\right)}}\right] =$

$$= -V_\infty \alpha \left[1 - \sqrt{1-\frac{c}{x}}\right]$$

- $\displaystyle v(x,y)\Big|_{x<0} = w(x) + V_\infty \cdot \alpha = V_\infty \alpha \sqrt{1-\frac{c}{x}}$

b) For $0 < x < c$ (along the airfoil):

- $A = \frac{2x}{c} - 1 \Rightarrow -1 \leq A \leq 1$

- $\displaystyle\int_{\theta=0}^{\theta=\pi} \frac{d\theta}{\cos\theta + A} = \int_{\theta=0}^{\theta=\pi} \frac{d\theta}{\cos\theta - \cos\theta_0} = \pi \cdot \frac{\sin(n\theta_0)}{\sin(\theta_0)}\Bigg|_{n=0} = 0$

- $\displaystyle w(x)\Big|_{0\leq x\leq c} = \frac{-V_\infty \cdot \alpha}{\pi}\left[\pi + 2\left(1-\frac{x}{c}\right)\cdot 0\right] = -V_\infty \cdot \alpha$

- $\displaystyle v(x,y)\Big|_{0\leq x\leq c} = w(x) + V_\infty \cdot \alpha = V_\infty \alpha(-1+1) = 0$

c) For $x > c$ (region after the trailing edge):

- $A \equiv \frac{2x}{c} - 1 > 1$

- $$\int_{\theta=0}^{\theta=\pi} \frac{d\theta}{\cos\theta + A} = \frac{\pi}{\sqrt{A^2-1}} = \frac{\pi}{2\sqrt{\frac{x}{c}\left(\frac{x}{c}-1\right)}}$$

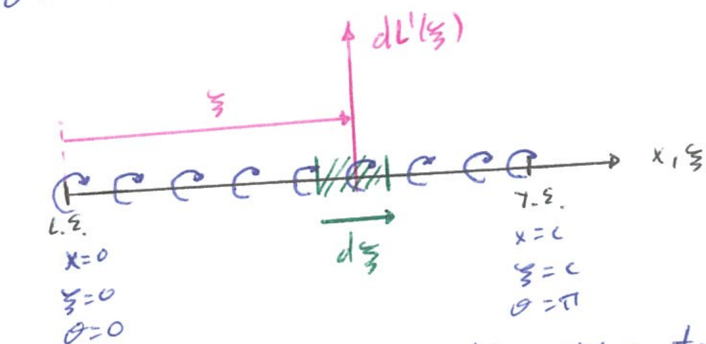
- $$w(x)\Big|_{x>c} = -\frac{V_\infty \alpha}{\pi}\left[\pi + \cancel{2}\left(1-\frac{x}{c}\right)\right)\frac{\pi}{\cancel{2}\sqrt{\frac{x}{c}\left(\frac{x}{c}-1\right)}}\Bigg] = -V_\infty \alpha\left[1 - \sqrt{1-\frac{c}{x}}\right]$$

- $$v(x,y)\Big|_{x>c} = w(x) + V_\infty \alpha = V_\infty \alpha\left[\cancel{-1} + \sqrt{1-\frac{c}{x}} + \cancel{1}\right] = V_\infty \alpha \sqrt{1-\frac{c}{x}}$$

So the self-induced velocity is:

$$\boxed{w(x) = -V_\infty \cdot \alpha \cdot \begin{cases} 1 & ; \text{ if } 0 \le \frac{x}{c} \le 1 \\ 1 - \sqrt{1-\frac{x}{c}} & ; \text{ otherwise} \end{cases}}$$

3) A diferential of the vortex sheet $d\xi$ generates a force $dL'(\xi)$:




where $dL'(\xi) = \rho_\infty V_\infty \gamma(\xi)\, d\xi$. The total lift is the integral along the chord.

$$* \; L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty \cdot \int_{\xi=0}^{\xi=c} \gamma(\xi)\, d\xi$$

The lift coefficient is, by definition:

$$* \; c_\ell = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 \cdot c} = \frac{2}{V_\infty \cdot c} \cdot \int_{\xi=0}^{\xi=c} \gamma(\xi)\, d\xi$$

By using the change of variable $\xi = \frac{c}{2}(1-\cos\theta)$:

$$* \int_{\xi=0}^{\xi=c} \gamma(\xi)\, d\xi = \int_{\theta=0}^{\theta=\pi} \frac{\cancel{2} V_\infty \cdot \alpha (1+\cos\theta)}{\cancel{\sin\theta}} \cdot \frac{c}{\cancel{2}} \cancel{\sin\theta}\, d\theta =$$

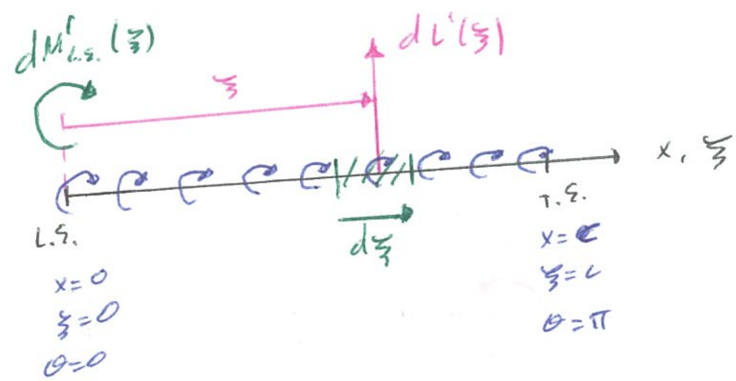
$$= V_\infty \cdot \alpha \cdot c \cdot \int_{\theta=0}^{\theta=\pi} (1+\cos\theta)\, d\theta = V_\infty \alpha c \left[\theta + \sin\theta\right]_{\theta=0}^{\theta=\pi} =$$

$$= V_\infty \cdot \alpha \cdot c \cdot \pi$$

And thus:

$$* \; \boxed{c_\ell = 2\pi\alpha}$$ $\Rightarrow$ This is the same result as the Homework 1 - Problem 2.

4) A differential of the vortex sheet $d\xi$ generates a force $dL'(\xi)$. This differential of force generates a moment $dM'_{L.E.}(\xi)$ about the leading edge:



where $dL'(\xi) = \rho_\infty V_\infty \gamma(\xi) d\xi$ and $dM'_{L.E.} = -\rho_\infty V_\infty \gamma(\xi)\cdot \xi d\xi$.

The total moment about the leading Edge is the integral along the chord:

$$* \quad M'_{L.E.} = \int_{\xi=0}^{\xi=c} dM'_{L.E.} = \int_{\xi=0}^{\xi=c} -\rho_\infty V_\infty \gamma(\xi)\cdot \xi \cdot d\xi$$

The moment coefficient is, by definition:

$$* \quad C_{m_{L.E.}} = \frac{M'_{L.E.}}{\frac{1}{2}\rho_\infty V_\infty^2 c^2} = \frac{-2}{V_\infty c^2}\int_{\xi=0}^{\xi=c} \gamma(\xi)\cdot \xi \cdot d\xi.$$

We use again the change of variable: $\xi = \frac{c}{2}(1-\cos\theta)$:

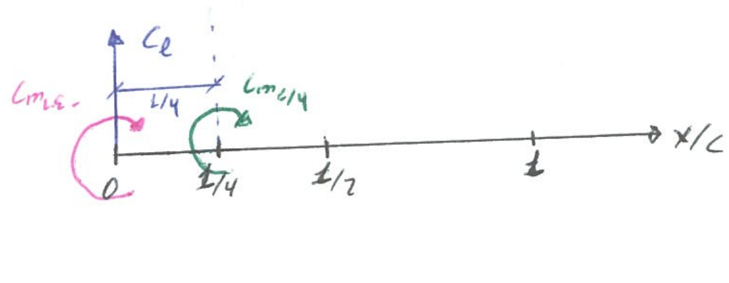
$$* \quad \int_{\xi=0}^{\xi=c} \gamma(\xi)\cdot\xi\cdot d\xi = \int_{\theta=0}^{\theta=\pi} \frac{\cancel{2} V_\infty \alpha (1+\cos\theta)}{\cancel{\sin\theta}}\cdot\frac{c}{\cancel{2}}(1-\cos\theta)\cdot\frac{c}{2}\cancel{\sin\theta}\, d\theta =$$

$$= \frac{V_\infty \alpha c^2}{2}\cdot\int_{\theta=0}^{\theta=\pi} \sin^2\theta\, d\theta = \frac{V_\infty \alpha c^2 \pi}{4}$$

And thus:

$$* \quad \boxed{C_{m_{L.E.}} = -\frac{\pi}{2}\alpha} \quad \Rightarrow$$ This is the same result as the Homework 1 - Problem 2.

We use the following moment balance to calculate $C_{m_{c/4}}$:



$* \quad C_{m_{c/4}} = C_{m_{L.E.}} + \frac{1}{4} \cdot C_l = -\frac{\pi}{2}\alpha + \frac{1}{4} \cdot 2\pi\alpha$

$* \quad \boxed{C_{m_{c/4}} = 0}$ $\Rightarrow$ This is the same result as the Homework 1 - Problem 2.

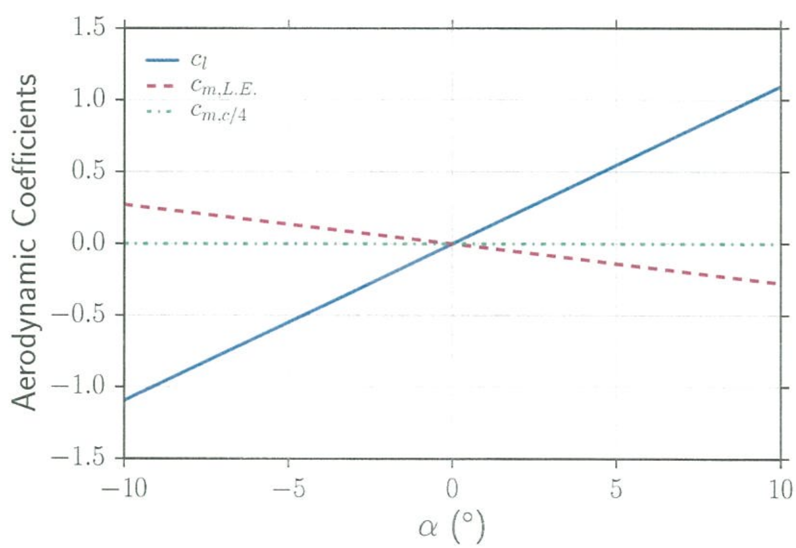
5) $* \quad C_l = \dfrac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = 2\pi\alpha \Rightarrow \alpha = \dfrac{L'}{\rho_\infty V_\infty^2 \cdot c \cdot \pi}$

$* \quad \boxed{\alpha \simeq 0.1\,rad \simeq 5.8^\circ}$

6) $* \quad C_{m_{L.E.}} = \dfrac{M'_{L.E.}}{\frac{1}{2}\rho_\infty V_\infty^2 c^2} = -\frac{\pi}{2}\alpha \Rightarrow \alpha = -\dfrac{4 M'_{L.E.}}{\rho_\infty V_\infty^2 c^2 \pi}$

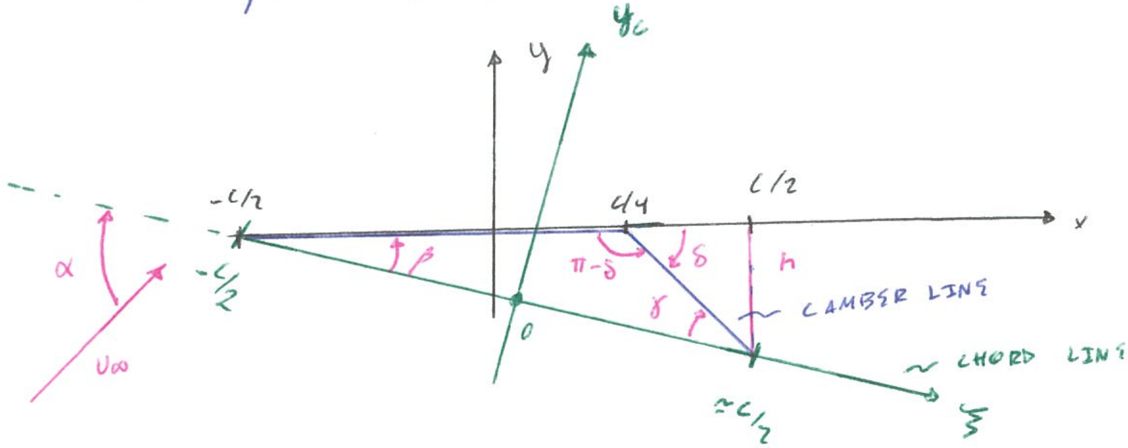
$* \quad \boxed{\alpha \simeq -0.06\,rad \simeq -3.3^\circ}$

7) The aerodynamic coefficients are shown in the following plot:



## Problem 2:

1) The camber line and the chord line present the following geometry:




2) It can be seen that:

* $\tan\delta = \frac{h}{c/4} \simeq \delta \Rightarrow \frac{h}{c} \simeq \frac{\delta}{4}$

By using this value, we can obtain $\beta$ as:

* $\tan\beta = \frac{h}{c} \simeq \beta \Rightarrow \beta \simeq \frac{\delta}{4}$

and $\gamma$ as:

* $\beta + (\pi - \delta) + \gamma = \pi \Rightarrow \gamma = \delta - \beta = \frac{3}{4}\delta$

The slope of the camber line is completely defined by $\beta$ and $\gamma$ as:

$$\boxed{\frac{dy_c}{dx} = \begin{cases} \delta/4 & ; -\frac{1}{2} \le \frac{x}{c} \le \frac{1}{4} \\ -3\delta/4 & ; \frac{1}{4} \le \frac{x}{c} \le \frac{1}{2} \end{cases}}$$

3) Make the change of variable $x = -\frac{c}{2} \cdot \cos\theta_0$ and get:

* $\frac{x}{c} = -\frac{1}{2} \Rightarrow \cos\theta_0 = 1 \Rightarrow \theta_0 = 0$

* $\frac{x}{c} = \frac{1}{4} \Rightarrow \cos\theta_0 = -\frac{1}{2} \Rightarrow \theta_0 = \frac{2}{3}\pi$

* $\frac{x}{c} = \frac{1}{2} \Rightarrow \cos\theta_0 = -1 \Rightarrow \theta_0 = \pi$

So:

$$\frac{dy_c(\theta_0)}{dx} = \begin{cases} \delta/4 & ; 0 \le \theta_0 \le \frac{2}{3}\pi \\ -3\delta/4 & ; \frac{2}{3}\pi \le \theta_0 \le \pi \end{cases}$$

We express it as the Fourier Series:

$$\frac{dy_c(\theta_0)}{dx} = (\alpha - A_0) + \sum_{n=1}^{n=\infty} A_n \cdot \cos(n\theta_0)$$

where:

$$A_0 = \alpha - \frac{1}{\pi} \int_{\theta=0}^{\theta=\pi} \frac{dy_c(\theta_0)}{dx} \, d\theta_0$$

$$A_n = \frac{2}{\pi} \int_{\theta=0}^{\theta=\pi} \frac{dy_c(\theta_0)}{dx} \cos(n\theta_0) \, d\theta_0$$

Calculating the integrals we get:

$$* \; A_0 = \alpha - \frac{1}{\pi} \cdot \left\{ \int_{\theta_0=0}^{\theta_0=2\pi/3} \frac{\delta}{4} \, d\theta_0 + \int_{\theta_0=2\pi/3}^{\theta_0=\pi} -\frac{3\delta}{4} \, d\theta_0 \right\} =$$

$$= \alpha - \frac{1}{\pi} \cdot \frac{\delta}{4} \cdot \left\{ \theta_0 \Big|_{\theta_0=0}^{\theta_0=2\pi/3} - 3\theta_0 \Big|_{\theta_0=2\pi/3}^{\theta_0=\pi} \right\} \Rightarrow \boxed{A_0 = \alpha + \frac{\delta}{12}}$$

$$* \; A_1 = \frac{2}{\pi} \left\{ \int_{\theta_0=0}^{\theta_0=2\pi/3} \frac{\delta}{4} \cos(\theta_0) \, d\theta_0 + \int_{\theta_0=2\pi/3}^{\theta_0=\pi} -\frac{3\delta}{4} \cos(\theta_0) \, d\theta_0 \right\} =$$

$$= \frac{2}{\pi} \cdot \frac{\delta}{4} \cdot \left\{ \sin\theta_0 \Big|_{\theta_0=0}^{\theta_0=2\pi/3} - 3\sin\theta_0 \Big|_{\theta_0=2\pi/3}^{\theta_0=\pi} \right\} \Rightarrow$$

$$\Rightarrow \boxed{A_1 = \frac{\sqrt{3}}{\pi} \delta}$$

$$* \; A_2 = \frac{2}{\pi} \left\{ \int_{\theta_0=0}^{\theta_0=2\pi/3} \frac{\delta}{4} \cos(2\theta_0) \, d\theta_0 + \int_{\theta_0=}^{} -\frac{3\delta}{4} \cos(2\theta_0) \, d\theta_0 \right\} =$$

$$= \frac{2}{4} \cdot \frac{\delta}{4} \cdot \left\{ \frac{1}{2} \sin(2\theta_0) \Big|_{\theta_0=0}^{\theta_0=2\pi/3} - \frac{3}{2} \sin(2\theta_0) \Big|_{\theta_0=2\pi/3}^{\theta_0=\pi} \right\} \Rightarrow$$

$$\Rightarrow \boxed{A_2 = -\frac{\sqrt{3}}{2\pi} \delta}$$

4)

$$* \; \boxed{\gamma(\theta) = 2V_\infty \left[ A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \cdot \sin(n\theta) \right]}$$

5)

* $\Gamma = V_\infty \pi c \left[A_0 + \frac{A_1}{2}\right] \Rightarrow \boxed{\Gamma = V_\infty \pi c \cdot \left[\alpha + \delta \cdot \left(\frac{1}{12} + \frac{\sqrt{3}}{2\pi}\right)\right]}$

6)

* $L' = \rho_\infty V_\infty \Gamma = \boxed{\rho_\infty V_\infty^2 \pi c \left[\alpha + \delta\left(\frac{1}{12} + \frac{\sqrt{3}}{2\pi}\right)\right] = L'}$

* $c_l = 2\pi A_0 + \pi A_1 = \boxed{2\pi\left[\alpha + \delta\left(\frac{1}{12} + \frac{\sqrt{3}}{2\pi}\right)\right] = c_l}$

* $\boxed{a_0 = 2\pi}$

* $\alpha_{L'=0} = \alpha - A_0 - \frac{A_1}{2} = \boxed{-\delta\left(\frac{1}{12} + \frac{\sqrt{3}}{2\pi}\right) = \alpha_{L'=0}}$

* $\alpha = \alpha_I \Rightarrow A_0 = 0 \Rightarrow \boxed{\alpha_I = -\frac{\delta}{12}}$

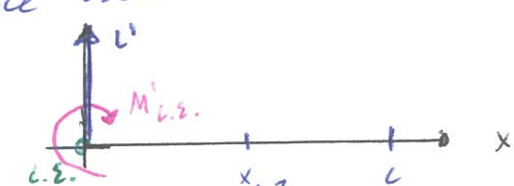
7)

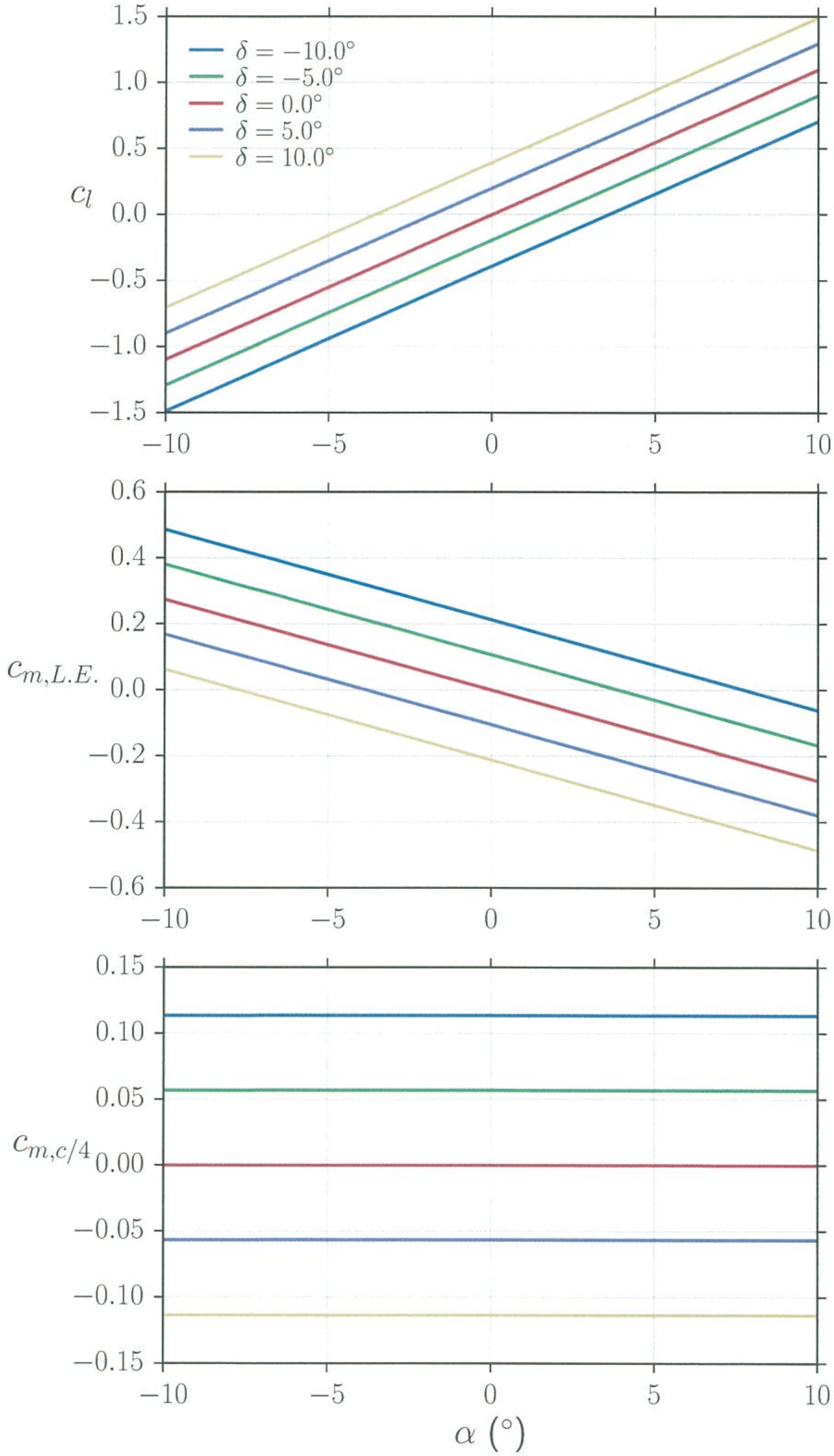
* $c_{m,L.E.} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right) = \boxed{-\frac{\pi}{2}\left[\alpha + \delta\left(\frac{1}{12} + \frac{5\sqrt{3}}{4\pi}\right)\right] = c_{m,L.E.}}$

* $c_{m,c/4} = -\frac{\pi}{4}\left(A_1 - A_2\right) = \boxed{-\frac{3\sqrt{3}}{8}\delta = c_{m,c/4}}$

* $\frac{x_{c.p.}}{c} = -\frac{c_{m,L.E.}}{c_l} \cdot c = \boxed{\frac{1}{4}\left[1 + \frac{\frac{3\sqrt{3}}{2\pi}\delta}{2\alpha + \left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right)\delta}\right] = \frac{x_{cP}}{c}}$

the moment balance used is:





$\delta = -10.0°$
$\delta = -5.0°$
$\delta = 0.0°$
$\delta = 5.0°$
$\delta = 10.0°$
$c_l$
$c_{m,L.E.}$
$c_{m,c/4}$
$\alpha$ (°)