MAE 104 - HOMEWORK #4:

Problem 1:

1)

* $G_b(\theta) = \frac{\Gamma(\theta)}{b V_\infty} = \frac{2 \cancel{V_\infty} \cancel{b} \cdot \sum_{n=1}^{n=\infty} A_n \cdot \sin(n\theta)}{\cancel{b} \cancel{V_\infty}} = 2 \sum_{n=1}^{n=\infty} A_n \cdot \sin(n\theta) =$

$= -\frac{6\varepsilon}{\pi \Lambda} \sin\theta - \frac{2\varepsilon}{\pi\Lambda} \sin(3\theta) \Rightarrow$

$A_1 = -\frac{3\cdot\varepsilon}{\pi\Lambda}$

$A_2 = 0$

$A_3 = -\frac{\varepsilon}{\pi\Lambda}$

$A_n = 0; \ \forall n > 3$

* $C_L = \pi \Lambda A_1 = \cancel{\pi} \cancel{\Lambda} \cdot \frac{-3\varepsilon}{\cancel{\pi}\cancel{\Lambda}} \Rightarrow \boxed{C_L = -3\varepsilon}$

2)

* $C_{D,i} = \pi \Lambda \sum_{n=1}^{n=\infty} n A_n^2 = \pi\Lambda \left[1 \cdot A_1^2 + 3A_3^2\right] = \pi\Lambda\left[\left(-\frac{3\varepsilon}{\pi\Lambda}\right)^2 + 3\left(-\frac{\varepsilon}{\pi\Lambda}\right)^2\right] =$

$= \cancel{\pi}\cancel{\Lambda} \frac{\varepsilon^2}{\pi^2\Lambda^2} \cdot [9+3] = \frac{\varepsilon^2}{\pi\Lambda} \cdot 12 \Rightarrow \boxed{C_{D,i} = \frac{6}{5\pi}\varepsilon^2}$

3)

* For each section θ:

• $C_\ell(\theta) = 2\pi\left[\alpha_{eff}(\theta) - \alpha_{l=0}(\theta)\right] = 2\pi\left[\alpha(\theta) - \alpha_i(\theta) - \alpha_{\ell=0}(\theta)\right]$

* We can calculate $\alpha(\theta) - \alpha_{\ell=0}(\theta)$ from the Fundamental Equation of Lifting Line Theory:

• $\alpha(\theta) - \alpha_{l=0}(\theta) = \frac{2b}{\pi \cdot c(\theta)} \cdot \sum_{n=1}^{n=\infty} A_n \cdot \sin(n\theta) + \sum_{n=1}^{n=\infty} n \cdot A_n \cdot \frac{\sin(n\theta)}{\sin(\theta)}$

* For an elliptical airfoil:

• $c(\theta) = c_0 \cdot \sin\theta$

• $S = \pi \cdot \frac{b}{2} \cdot \frac{c_0}{2} \Rightarrow \Lambda = \frac{b^2}{S} = \frac{4b}{\pi \cdot c_0}$

So:

- $\alpha(\theta) - \alpha_{l=0}(\theta) = \sum_{n=1}^{n=\infty} \left(\frac{\Lambda}{2} + n\right) A_n \cdot \frac{\sin(n\theta)}{\sin(\theta)}$

* Plugging the values of A_n:

- $\alpha(\theta) - \alpha_{l=0}(\theta) = \left(\frac{\Lambda}{2} + 1\right) A_1 + \left(\frac{\Lambda}{2} + 3\right) \frac{\sin(3\theta)}{\sin(\theta)}$

* We can also calculate the induced angle of attack as:

- $\alpha_i(\theta) = \sum_{n=1}^{n=\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)} = A_1 + 3A_3 \frac{\sin(3\theta)}{\sin\theta}$

So:

- $c_l(\theta) = 2\pi\left[\alpha(\theta) - \alpha_{l=0}(\theta) - \alpha_i(\theta)\right] =$

$$= 2\pi\left[\left(\frac{\Lambda}{2} + \cancel{1}\right) A_1 + \left(\frac{\Lambda}{2} + \cancel{3}\right) A_3 \frac{\sin(3\theta)}{\sin\theta} - \cancel{A_1} - \cancel{3A_3 \frac{\sin(3\theta)}{\sin(\theta)}}\right] =$$

$$= \cancel{2}\pi \frac{\Lambda}{\cancel{2}} \cdot \left[A_1 + A_3 \cdot \frac{\sin(3\theta)}{\sin(\theta)}\right] =$$

$$= \cancel{\pi\Lambda} \cdot \left[\frac{-3\varepsilon}{\cancel{\pi\Lambda}} - \frac{\varepsilon}{\cancel{\pi\Lambda}} \frac{\sin(3\theta)}{\sin(\theta)}\right] =$$

$$= -\varepsilon\left[3 + \frac{\sin(3\theta)}{\sin(\theta)}\right]$$

* We simplify this formula by using the trigonometric relations:

- $\sin(3\theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin(\theta)$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta =$

$$= 3\sin\theta - 4\sin^3\theta$$

and obtain:

- $\boxed{C_\ell(\theta) = -2\varepsilon\left[3 - 2\sin^2\theta\right]}$

4) * The stall begins in the section where $C_\ell(\theta)$ is maximum, i.e.:

- $\dfrac{d\,C_\ell(\theta)}{d\theta} = 0$
- $\dfrac{d^2 C_\ell(\theta)}{d\theta^2} < 0$

* In this problem:

- $\dfrac{d\,C_\ell(\theta)}{d\theta} = 8\cdot\varepsilon\cdot\sin\theta\cdot\cos\theta = 0 \Rightarrow \begin{cases} \theta = 0, \pi \\ \theta = \pi/2 \end{cases}$

- $\dfrac{d^2 C_\ell(\theta)}{d\theta} = 8\,\varepsilon\cdot(\cos^2\theta - \sin^2\theta)$

- We check both solutions:

  * $\left.\dfrac{d^2 C_\ell}{d\theta}\right|_{\theta=0,\pi} = 8\,\varepsilon\cdot(\cancelto{1}{\cos^2 0} - \cancelto{0}{\sin^2 0}) = 8\,\varepsilon > 0 \Rightarrow$

    $\Rightarrow$ It is a minimum (if $\varepsilon > 0$).

    $\left.\dfrac{d^2 C_\ell}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 8\cdot\varepsilon\cdot\left[\cancelto{0}{\cos^2\left(\tfrac{\pi}{2}\right)} - \cancelto{1}{\sin^2\left(\tfrac{\pi}{2}\right)}\right] = -8\,\varepsilon < 0 \Rightarrow$

    $\Rightarrow$ It is a maximum (if $\varepsilon < 0$).

* So:

- If $\varepsilon > 0$, the stall begins at $\theta = \frac{\pi}{2}$ (root section).
- If $\varepsilon < 0$, the stall begins at $\theta = 0, \pi$ (wing tips).

Problem 2:

1) * We make the change of variable: $\frac{y}{b_0} = \frac{1}{2}\cos\theta_0$.

* The expression for $x_{l.e.}(\theta_0)$ and $x_{t.e.}(\theta_0)$ takes the form:

- $\frac{x_{l.e.}(\theta_0)}{c_0} = -\frac{1}{4}\sqrt{1-\cos^2\theta_0} = -\frac{1}{4}\sin\theta_0$

- $\frac{x_{t.e.}(\theta_0)}{c_0} = \frac{3}{4}\sqrt{1-\cos^2\theta_0} = \frac{3}{4}\sin\theta_0$

* By definition, the chord is:

- $c(\theta_0) = x_{l.e.}(\theta_0) - x_{t.e.}(\theta_0) = c_0 \cdot \sin\theta_0$

* The non-dimensional chord is, then:

- $K(\theta_0) = \frac{c(\theta_0)}{b_0} = \frac{c_0}{b_0}\cdot\sin\theta_0$

* The area of an ellipse is:

- $S = \frac{\pi}{4}\cdot b_0 \cdot c_0$

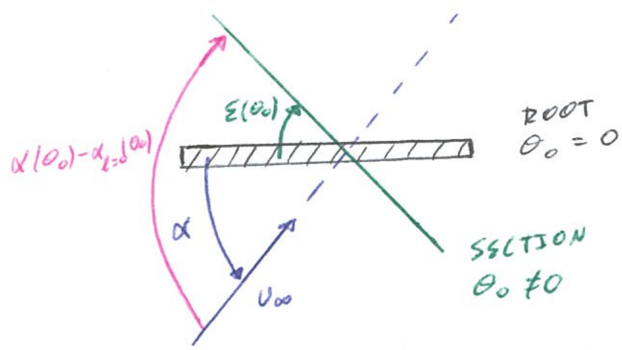
and by definition, the aspect ratio is:

- $\Lambda = \frac{b_0^2}{S} = \frac{4 b_0}{\pi c_0} \Rightarrow \frac{c_0}{b_0} = \frac{4}{\pi\Lambda}$

* According to this:

- $\boxed{K(\theta_0) = \frac{4}{\pi\Lambda}\cdot\sin\theta_0}$

2) * In the following sketch, I include the free stream, the root section and a secton θ_0.



* We know that, for a flat plate: $\alpha_{l=0} = 0$

* And given the geometry of the problem:

$$\boxed{\alpha(\theta_0) - \alpha_{l=0}(\theta_0) = \alpha + \varepsilon(\theta_0)}$$

* We use the change of variable $\frac{y}{b_0} = \frac{1}{2}\cos\theta_0$:

$$\varepsilon(y) = \varepsilon_0 \cdot \frac{|y|}{b_0} \Rightarrow \varepsilon(\theta_0) = \begin{cases} -\varepsilon_0 \cdot \cos\theta_0; & \frac{\pi}{2} \leq \theta_0 \leq \pi \\ \varepsilon_0 \cdot \cos\theta_0; & 0 \leq \theta_0 \leq \frac{\pi}{2} \end{cases}$$

3) * The Fundamental Equation of Lifting Line Theory is:

$$\alpha(\theta_0) = \frac{2}{\pi\,\mathcal{R}(\theta_0)} \cdot \sum_{n=1}^{n=\infty} A_n \cdot \sin(n\theta_0) + \alpha_{l=0}(\theta_0) + \sum_{n=1}^{n=\infty} n A_n \frac{\sin(n\theta_0)}{\sin(\theta_0)}$$

* We multiply this equation by $\frac{\pi}{2}\cdot\mathcal{R}(0)$ and substitute the results from the previous parts:

$$\boxed{\frac{2}{\Lambda}\left[\alpha\cdot\sin\theta_0 + \varepsilon(\theta_0)\cdot\sin\theta_0\right] = \sum_{n=1}^{n=\infty}\left(1+\frac{2n}{\Lambda}\right)A_n\sin(n\theta_0)} \quad \text{(I)}$$

4)

* In order to obtain the coefficients A_n, we have to express $[\alpha \cdot \sin\theta_0 + \varepsilon(\theta_0) \cdot \sin\theta_0]$ as a series of sines:

- $[\alpha \cdot \sin\theta_0 + \varepsilon(\theta_0) \cdot \sin\theta_0] = \sum_{n=1}^{n=\infty} B_n \cdot \sin(n\theta_0)$ (II)

- $$B_1 = \frac{2}{\pi} \cdot \int_{\theta_0=0}^{\theta_0=\pi} [\alpha + \varepsilon(\theta_0)] \sin\theta_0 \sin\theta_0 \, d\theta_0 =$$

$$= \frac{2}{\pi} \cdot \left\{ \alpha \cdot \int_{\theta_0=0}^{\theta_0=\pi} \sin^2\theta_0 \, d\theta_0 + \varepsilon_0 \cdot \int_{\theta_0=0}^{\theta_0=\pi/2} \cos\theta_0 \sin^2\theta_0 \, d\theta_0 + \varepsilon_0 \cdot \int_{\theta_0=\pi/2}^{\theta_0=\pi} -\cos\theta_0 \sin^2\theta_0 \, d\theta_0 \right\} =$$

$$= \frac{2}{\pi} \left\{ \alpha \cdot \frac{\pi}{2} + \varepsilon_0 \cdot \int_{\theta_0=0}^{\theta_0=\pi/2} \sin^2\theta_0 \, d[\sin\theta_0] - \varepsilon_0 \cdot \int_{\theta_0=\pi/2}^{\theta_0=\pi} \sin^2\theta_0 \, d[\sin\theta_0] \right\} =$$

$$= \frac{2}{\pi} \left\{ \alpha \frac{\pi}{2} + \varepsilon_0 \left. \frac{\sin^3\theta_0}{3} \right|_{\theta_0=0}^{\theta=\pi/2} - \varepsilon_0 \left. \frac{\sin^3\theta_0}{3} \right|_{\theta_0=\pi/2}^{\theta_0=\pi} \right\} =$$

$$= \frac{2}{\pi} \left\{ \alpha \cdot \frac{\pi}{2} + \varepsilon_0 \cdot \frac{1}{3} + \varepsilon_0 \cdot \frac{1}{3} \right\} = \alpha + \frac{4}{3\pi} \varepsilon_0$$

- $$B_n = \frac{2}{\pi} \int_{\theta_0=0}^{\theta_0=\pi} [\alpha + \varepsilon(\theta_0)] \sin\theta_0 \sin(n\theta_0) \, d\theta_0 =$$

$$= \frac{2}{\pi} \left\{ \alpha \underbrace{\int_{\theta_0=0}^{\theta_0=\pi} \sin\theta_0 \sin(n\theta_0) \, d\theta_0}_{0} + \int_{\theta_0=0}^{\theta_0=\pi} \varepsilon(\theta_0) \sin(n\theta_0) \sin\theta_0 \, d\theta_0 \right\} =$$

FOR $n \neq 1$

$$= \frac{2}{\pi} \cdot \varepsilon_0 \cdot \left\{ \int_{\theta_0=0}^{\theta_0=\pi/2} \sin\theta_0 \cos\theta_0 \sin(n\theta_0) \, d\theta_0 + \int_{\theta_0=\pi/2}^{\theta_0=\pi} -\sin\theta_0 \cos\theta_0 \sin(n\theta_0) \, d\theta_0 \right\} =$$

$$= \frac{\cancel{2}}{\pi} \mathcal{E}_0 \cdot \left\{ \int_{\theta_0=0}^{\theta_0=\pi/2} \frac{\sin(2\theta_0)}{\cancel{2}} \sin(n\theta_0)\, d\theta_0 - \int_{\theta_0=\pi/2}^{\theta_0=\pi} \frac{\sin(2\theta_0)}{\cancel{2}} \sin(n\theta_0)\, d\theta_0 \right\} =$$

$$= \frac{\mathcal{E}_0}{\pi} \cdot \left\{ \int_{\theta_0=0}^{\theta_0=\pi/2} \left[\cos((2-n)\theta_0) - \cos((2+n)\theta_0) \right] d\theta_0 - \right.$$

$$\left. - \int_{\theta_0=\pi/2}^{\theta_0=\pi} \left[\cos((2-n)\theta_0) - \cos((2+n)\theta_0) \right] d\theta_0 \right\} =$$

$$= \frac{\mathcal{E}_0}{\pi} \cdot \left\{ \left[\frac{\sin[(2-n)\theta_0]}{2-n} - \frac{\sin[(2+n)\theta_0]}{2+n} \right]_{\theta_0=0}^{\theta_0=\pi/2} - \left[\frac{\sin[(2-n)\theta_0]}{2-n} - \frac{\sin[(2+n)\theta_0]}{2+n} \right]_{\theta_0=\pi/2}^{\theta_0=\pi} \right\}$$

$$= \frac{\mathcal{E}_0}{\pi} \left\{ - \frac{2 \sin\left(\frac{\pi n}{2}\right)}{n^2-4} - \frac{2\left[\sin\left(\frac{n\pi}{2}\right) + \sin(\pi n)\right]}{n^2-4} \right\} =$$

$$= \frac{-2\mathcal{E}_0}{\pi(n^2-4)} \left[\sin(n\pi) + 2 \cdot \sin\left(n\frac{\pi}{2}\right) \right]$$

* We can see that:
  - ⓐ when n is even: $\sin(n\pi) = \sin\left(\frac{\pi n}{2}\right) = 0 \Rightarrow B_n = 0$
  - ⓑ when n is odd:
    * $n = 2k+1$
    * $\sin(n\pi) = 0$
    * $\sin\left(\frac{2k+1}{2} \cdot \pi\right) = \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k$
    * $B_{n=2k+1} = \frac{-2\mathcal{E}_0}{\pi(n^2-4)} \left[0 + 2 \cdot (-1)^k \right] = \frac{-4\mathcal{E}_0}{\pi(n^2-4)} (-1)^k$
    * $B_{2k+1} \sim \frac{1}{n^2}$
    * $B_{2k+1} \neq 0$

* As a summary:

- $B_1 = \alpha + \frac{4}{3\pi}\varepsilon_0$
- $B_{2k} = 0$
- $B_{2k+1} = \frac{-\varepsilon_0 \cdot (-1)^k}{\pi\left(k^2+k-\frac{3}{4}\right)} = \frac{-\varepsilon_0 (-1)^k}{\pi\left(k-\frac{1}{2}\right)\left(k+\frac{3}{2}\right)}$

* By equating equations I and II we obtain:

- $\frac{2}{\Lambda}\sum_{n=1}^{n=\infty} B_n \cdot \sin(n\theta_0) = \sum_{n=1}^{n=\infty}\left(1+\frac{2n}{\Lambda}\right)A_n \cdot \sin(n\theta_0) \Rightarrow$

$$\Rightarrow \frac{2}{\Lambda}B_n = \left(1+\frac{2n}{\Lambda}\right)A_n$$

- $A_n = \frac{B_n}{n+\frac{\Lambda}{2}}$

* By usin the values of B_n:

$$\boxed{\begin{aligned} & A_1 = \frac{\alpha + \frac{4}{3\pi}\varepsilon_0}{1+\frac{\Lambda}{2}} \\ & A_{2k} = 0 \\ & A_{2k+1} = \frac{-\varepsilon_0 \cdot (-1)^k}{\pi\left(2k+1+\frac{\Lambda}{2}\right)\left(k-\frac{1}{2}\right)\left(k+\frac{3}{2}\right)} \quad ; k>0 \end{aligned}}$$

5)

* We calculate C_L as:

$$\boxed{C_L = \pi \Lambda A_1 = \pi \cdot \frac{\alpha + \frac{4\varepsilon_0}{3\pi}}{\frac{1}{2} + \frac{1}{\Lambda}}}$$

* And L as:

$$L = \frac{1}{2}\rho U_\infty^2 S \cdot C_L$$

$$\boxed{L = \rho U_\infty^2 b_0^2 \frac{\alpha \cdot \pi + 4\varepsilon_0/3}{\Lambda + 2}}$$

6)

* The induced drag coefficient has the form: $n = 2k+1$

$$C_{D,i} = \pi \Lambda \sum_{n=1}^{n=\infty} n A_n^2 = \pi \Lambda A_1^2 + \pi \Lambda \sum_{n=2}^{n=\infty} n A_n^2 =$$

$$= \pi \Lambda \left(\frac{\alpha + \frac{4\varepsilon_0}{3\pi}}{1 + \Lambda/2}\right)^2 + \pi \Lambda \sum_{k=1}^{k=\infty} (2k+1) A_{2k+1}^2$$

$$\boxed{C_{D,i} = \pi \Lambda \cdot \left\{ \left(\frac{\alpha + \frac{4\varepsilon_0}{3\pi}}{1 + \Lambda/2}\right)^2 + \varepsilon_0 \sum_{k=1}^{k=\infty} \frac{(-1)^{2k}}{\left(\frac{\Lambda}{2} + 2k + 1\right)^2} \cdot \frac{2k+1}{\left(k - \frac{1}{2}\right)^2 \left(k + \frac{3}{2}\right)^2} \right\}}$$

* And the induced drag on the wing is:

$$\boxed{D_i = \frac{1}{2}\rho U_\infty^2 S C_{D,i} = \frac{1}{2}\rho U_\infty^2 \frac{b_0^2}{\Lambda} C_{D,i}}$$