Problem 1:

a)

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{1+t} + \frac{1}{1+2t} + 0 \neq 0$$

$$\boxed{\nabla\cdot\vec{V} = \frac{1}{1+t} + \frac{1}{1+2t}}$$

why is $\nabla\cdot\vec{V}\neq 0$? Because the fluid is compre-ssible and non-steady, and the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

$$\nabla\wedge\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{Bmatrix} = \vec{0}$$

$$\boxed{\nabla\wedge\vec{V} = \vec{0}}$$

If the flow is irrotational, then there is a potential function.

$$\vec{V} = \nabla\phi \Rightarrow u = \frac{\partial\phi}{\partial x} = \frac{x}{1+t} \Rightarrow \phi = \frac{1}{1+t}\cdot\frac{x^2}{2} + f(y,z,t)$$

$$v = \frac{\partial\phi}{\partial y} = \frac{\partial f}{\partial y} = \frac{y}{1+2t} \Rightarrow f = \frac{1}{2}\frac{y^2}{1+2t} + g(z,t)$$

$$w = \frac{\partial\phi}{\partial z} = \frac{\partial g}{\partial z} = 0 \Rightarrow g = C_1(t)$$

$$\boxed{\phi = \frac{1}{1+t}\cdot\frac{x^2}{2} + \frac{1}{1+2t}\frac{y^2}{2} + C_1(t)}$$

b) <u>Streamlines:</u> $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$$\frac{dx}{u} = \frac{dx}{x/1+t} = \frac{dy}{y/1+2t} = \frac{dy}{v} \Rightarrow \frac{dy}{y} = \frac{1+t}{1+2t} \cdot \frac{dx}{x} \Rightarrow$$

$$\Rightarrow \ln(y) = \ln(x)^{1+t/1+2t} + \ln C_1(t) \Rightarrow$$

$$\Rightarrow \boxed{y = C_1(t) \cdot x^{1+t/1+2t}}$$

$$\frac{dz}{w} = \frac{dy}{v} \Rightarrow dz = 0 \Rightarrow \boxed{z = cte}$$

<u>Trajectories:</u> $\vec{x}(t = t_0) = \vec{x}_0 = (x_0, y_0)$

$$\frac{dx}{dt} = u = \frac{x}{1+t} \Rightarrow \frac{dx}{x} = \frac{dt}{1+t} \Rightarrow \ln(x) = \ln(1+t) + \ln(C_1) \Rightarrow$$

$$\Rightarrow x = C_1 \cdot (1+t)$$

$$\frac{dy}{dt} = v = \frac{y}{1+2t} \Rightarrow \frac{dy}{y} = \frac{dt}{1+2t} \Rightarrow \ln(y) = \frac{1}{2}\ln(1+2t) + \ln(C_2) \Rightarrow$$

$$\Rightarrow y = C_2 \cdot (1+2t)^{1/2}$$

$$\frac{dz}{dt} = w = 0 \Rightarrow z = C_3$$

$$t = t_0 \Rightarrow \left| \begin{array}{l} x_0 = C_1 \\ y_0 = C_2 \\ z_0 = C_3 \end{array} \right| \Rightarrow \left\{ \boxed{\begin{array}{l} x = x_0(1+t) \\ y = y_0(1+2t)^{1/2} \\ z = z_0 \end{array}} \right.$$

<u>Paths:</u> Eliminate $t$

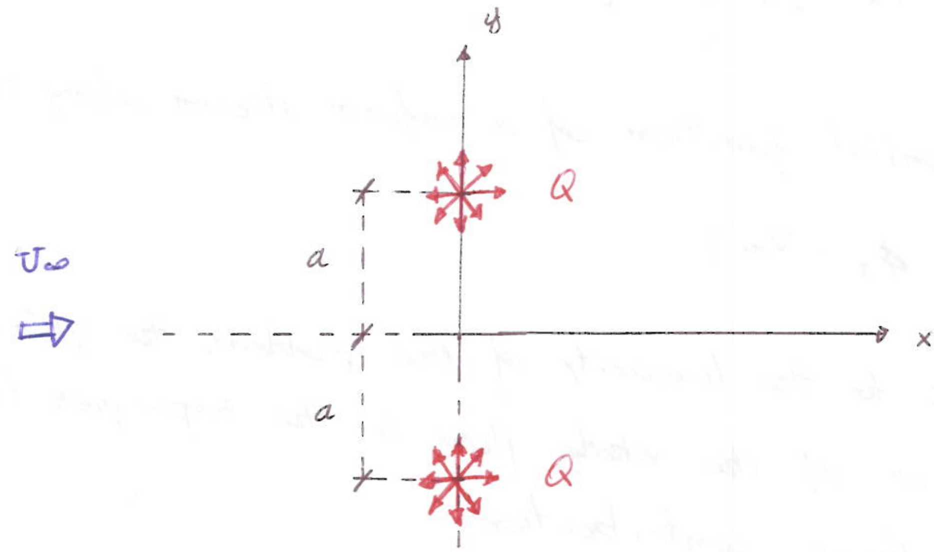
$$t = \frac{x}{x_0} - 1 \Rightarrow \boxed{\begin{array}{l} y = y_0\left(\frac{2x}{x_0} - 1\right)^{1/2} \\ z = z_0 \end{array}}$$

c) <u>streaklines:</u> fluid particles that in previous times ($t=\tau$) passed by the point $\vec{x}_p = (x_p, y_p)$

$$x_p = c_1 \cdot (1+\tau) \Rightarrow c_1 = \frac{x_p}{1+\tau} \Rightarrow \boxed{x = x_p \cdot \frac{1+t}{1+\tau}}$$

$$y_p = c_2 \cdot (1+2\tau)^{1/2} \Rightarrow c_2 = \frac{y_p}{(1+2\tau)^{1/2}} \Rightarrow \boxed{y = y_p \cdot \left(\frac{1+2t}{1+2\tau}\right)^{1/2}}$$

Problem 2:




1)

* In order to calculate the potential function for the upper source $\phi_1$, we follow these steps:

- Potential function for a source at the origin in polar coordinates:

$$\phi_o = \frac{Q}{2\pi} \ln\left(\frac{r}{a}\right)$$

- In cartesian coordinates:

$$\phi_0 = \frac{Q}{2\pi} \ln\left[\frac{(x^2+y^2)^{1/2}}{a}\right] = \frac{Q}{2\pi}\frac{1}{2} \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2\right]$$

- translate the source to $(x,y) = (0,a)$:

$$\phi_1 = \frac{Q}{2\pi}\frac{1}{2} \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} - 1\right)^2\right]$$

* Repeat with the source located at $(x,y)=(0,-a)$:

$$\phi_2 = \frac{Q}{2\pi}\,\frac{1}{2}\ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2\right]$$

* Potential function of a uniform stream along the x-axis:

$$\phi_3 = U_\infty x$$

* Due to the linearity of the problem, the potential function of the whole flow is the superposition of the three contributions:

$$\boxed{\phi(x,y) = U_\infty x + \frac{Q}{2\pi}\cdot\frac{1}{2}\left\{\ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2\right]+\ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2\right]\right\}}$$

2)

* $u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y} = U_\infty + \dfrac{Q}{2\pi}\,\dfrac{1}{2}\left\{\dfrac{1}{a}\,\dfrac{2x/a}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2}+\dfrac{1}{a}\,\dfrac{2x/a}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2}\right\} \Rightarrow$

$$\Rightarrow \boxed{u(x,y) = U_\infty + \frac{Q}{2\pi}\left\{\frac{x}{x^2+(y-a)^2}+\frac{x}{x^2+(y+a)^2}\right\}}$$

* $v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x} = \dfrac{Q}{2\pi}\,\dfrac{1}{2}\left\{\dfrac{1}{a}\,\dfrac{2\left(\frac{y}{a}-1\right)}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2}+\dfrac{1}{a}\,\dfrac{2\left(\frac{y}{a}+1\right)^2}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2}\right\} \Rightarrow$

$$\Rightarrow \boxed{v(x,y) = \frac{Q}{2\pi}\left\{\frac{y-a}{x^2+(y-a)^2}+\frac{y+a}{x^2+(y+a)^2}\right\}}$$

3) The compatibility conditions relate the streamfunction and the potential function:

$$
* \begin{cases} u = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x} \\[2ex] v = -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial y} \end{cases}
$$

Applying the first:

$$
* \; u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \Rightarrow \psi = \int u \, dy = U_\infty y + \frac{Q}{2\pi}\left\{ \operatorname{atan}\left(\frac{y-a}{x}\right) + \operatorname{atan}\left(\frac{y+a}{x}\right)\right\} + F(x)
$$

We need to calculate the free function $F(x)$. We can do it by plugging $\psi(x,y)$ in the second compatibility condition:

$$
* \; v = \frac{Q}{2\pi}\left\{ \frac{y-a}{x^2+(y-a)^2} + \frac{y+a}{x^2+(y+a)^2} \right\} = -\frac{\partial \psi}{\partial x} = \frac{Q}{2\pi}\left\{ \frac{y-a}{x^2+(y-a)^2} + \frac{y+a}{x^2+(y+a)^2} \right\} - F'(x) \Rightarrow
$$

$$
\Rightarrow F'(x) = 0 \Rightarrow F(x) = cte.
$$

As expected, the streamfunction (and the potential function) are defined up to a free constant. We can choose any value that we want. I choose $F(y)=0$ because it makes things simpler:

$$
* \; \boxed{\psi(x,y) = U_\infty y + \frac{Q}{2\pi}\left\{ \operatorname{atan}\left(\frac{y-a}{x}\right) + \operatorname{atan}\left(\frac{y+a}{x}\right)\right\}}
$$

4)

$$* \quad \frac{u}{Q/2\pi a} = \frac{U_\infty}{Q/2\pi a} + \frac{Q/2\pi}{Q/2\pi a}\left\{\frac{1}{a}\frac{r\cos\theta/a}{\left(\frac{r\cos\theta}{a}\right)^2+\left(\frac{r\sin\theta}{a}-1\right)^2} + \frac{1}{a}\frac{r\cos\theta/a}{\left(\frac{r\cos\theta}{a}\right)^2+\left(\frac{r\sin\theta}{a}+1\right)^2}\right\} =$$

$$= K + \frac{R\cos\theta}{R^2\cos^2\theta + (R\sin\theta-1)^2} + \frac{R\cos\theta}{R^2\cos^2\theta+(R\sin\theta+1)^2} =$$

$$= K + R\cos\theta\left\{\frac{1}{R^2+2R\sin\theta+1} + \frac{1}{R^2-2R\sin\theta+1}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\frac{u(R,\theta)}{Q/2\pi a} = K + \frac{2R\cos\theta\,(R^2+1)}{(R^2+1)^2-4R^2\sin^2\theta}}$$

$$* \quad \frac{v}{Q/2\pi a} = \frac{Q/2\pi}{Q/2\pi a}\left\{\frac{1}{a}\frac{\frac{r\sin\theta}{a}-1}{\left(\frac{r\cos\theta}{a}\right)^2+\left(\frac{r\sin\theta}{a}-1\right)^2} + \frac{1}{a}\frac{\frac{r\sin\theta}{a}+1}{\left(\frac{r\cos\theta}{a}\right)^2+\left(\frac{r\sin\theta}{a}+1\right)^2}\right\} =$$

$$= \frac{R\sin\theta-1}{R^2\cos^2\theta+(R\sin\theta-1)^2} + \frac{R\sin\theta+1}{R^2\cos^2\theta+(R\sin\theta+1)^2} =$$

$$= \frac{R\sin\theta-1}{R^2-2R\sin\theta+1} + \frac{R\sin\theta+1}{R^2+2R\sin\theta+1} \Rightarrow$$

$$\Rightarrow \boxed{\frac{v(R,\theta)}{Q/2\pi a} = \frac{2R\sin\theta\,(R^2-1)}{(R^2+1)^2-4R^2\sin^2\theta}}$$

5)

$$* \quad v(R,\theta)=0 \Rightarrow 2R\sin\theta\,(R^2-1)=0 \Rightarrow \left|\begin{array}{l} \boxed{R=0} \\ R^2-1=0 \Rightarrow \boxed{R=1} \\ \sin\theta=0 \Rightarrow \left|\begin{array}{l}\boxed{\theta=0}\\ \boxed{\theta=\pi}\end{array}\right. \end{array}\right.$$

6)

* $R=0$:

• $u(R=0,\theta)=0 \Rightarrow K=0 \Rightarrow$ Only valid if $U_\infty=0$

* $R=1$:

• $u(R=1,\theta)=0 \Rightarrow K+\dfrac{2\cos\theta\cdot 2}{4-4\sin^2\theta}=K+\dfrac{\cos\theta}{1-\sin^2\theta}=K+\dfrac{\cos\theta}{\cos^2\theta}=$

$$=K+\frac{1}{\cos\theta}=0 \Rightarrow \cos\theta=\frac{-1}{K}$$

$\Rightarrow K>1 \Rightarrow \cos\theta=\dfrac{-1}{K} \Rightarrow \theta=\operatorname{acos}\left(\dfrac{-1}{K}\right)+(2n+1)\cdot\pi$

$\Rightarrow K=1 \Rightarrow \cos\theta=-1 \Rightarrow \theta=\operatorname{acos}(-1)=(2n+1)\cdot\pi$

$\Rightarrow K<1 \Rightarrow \cos\theta=\dfrac{-1}{K}<-1 \Rightarrow \nexists$ solution

* $\theta=0$:

• $u(R,\theta=0)=0 \Rightarrow K+\dfrac{2R(R^2+1)}{(R^2+1)^2}=K+\dfrac{2R}{R^2+1}=0 \Rightarrow$

$\Rightarrow KR^2+2R+K=0 \Rightarrow R=\dfrac{-2\pm\sqrt{4-4K^2}}{2K}=\dfrac{-1}{K}\pm\sqrt{\dfrac{1}{K^2}-1}$

Need $R\geq 0 \Rightarrow R=-\dfrac{1}{K}+\sqrt{\dfrac{1}{K^2}-1}$

$\Rightarrow K>1 \Rightarrow \dfrac{1}{K^2}-1<0 \Rightarrow$ No physical solution.

$\Rightarrow K=1 \Rightarrow R=-1 \Rightarrow$ No physical solution.

$\Rightarrow K<1 \Rightarrow R=\dfrac{-1}{K}+\sqrt{\dfrac{1}{K^2}-1}<0 \Rightarrow$ No physical solution.

* $\theta = \pi$:

- $u(R, \theta=\pi) = 0 \Rightarrow K + \dfrac{2R(-1)(R^2+1)}{(R^2+1)^2 - 4R^2 \cdot 0} = K - \dfrac{2R}{R^2+1} = 0 \Rightarrow$

$$\Rightarrow K R^2 - 2R + K = 0 \Rightarrow R = \frac{2 \pm \sqrt{4 - 4K^2}}{2K} = \frac{1}{K} \pm \sqrt{\frac{1}{K^2} - 1}$$

$\Rightarrow K > 1 \Rightarrow \frac{1}{K^2} - 1 < 0 \Rightarrow$ No physical solution.

$\Rightarrow K = 1 \Rightarrow R = 1 \pm \sqrt{0} = 1$

$\Rightarrow K < 1 \Rightarrow R = \frac{1}{K} \pm \sqrt{\frac{1}{K^2} - 1} > 0$

* Summary:

$\Rightarrow$ $K < 1$: two symple stagnation points.

- $(R, \theta)_1 = \left( \frac{1}{K} + \sqrt{\frac{1}{K^2} - 1}, \pi \right)$
- $(R, \theta)_2 = \left( \frac{1}{K} - \sqrt{\frac{1}{K^2} - 1}, \pi \right)$

$\Rightarrow$ $K = 1$: one double stagnation point.

- $(R, \theta)_1 = (1, \pi)$
- $(R, \theta)_2 = (1, \pi)$

$\Rightarrow$ $K > 1$:

- $(R, \theta)_1 = \left[ 1, \operatorname{acos}\left( \frac{-1}{K} \right) \right]$
- $(R, \theta)_2 = \left[ 1, -\operatorname{acos}\left( \frac{-1}{K} \right) + 2\pi \right]$

7)

* $\dfrac{\Psi(x,y)}{Q/2\pi} = K\dfrac{y}{a} + \left\{ atan\left(\dfrac{\frac{y}{a}-1}{x/a}\right) + atan\left(\dfrac{\frac{y}{a}+1}{x/a}\right) \right\}$

* $\underline{K < 1:}$

- Stagnation Point 1: $(R,\theta)_1 = (R_1,\pi) \Rightarrow (x,y)_1 = (\underbrace{-R_1 \cdot a}_{<0}, 0)$

* In every point of a streamline, the value of $\Psi(x,y)$ is constant. In the streamline that passes through the stagnation point 1, $\Psi_1(x,y)$, the value is:

- $\dfrac{\Psi_1(x_1,y_1)}{Q/2\pi} = atan\left(\dfrac{-1}{-R_1}\right) + atan\left(\dfrac{1}{-R_1}\right) = 0$

* When $x \to \pm\infty$:

- $\lim\limits_{x\to\pm\infty} \left[ atan\left(\dfrac{y/a - 1}{x/a}\right)\right] = \pm n\dfrac{\pi}{2}$

- $\lim\limits_{x\to\pm\infty} \left[\dfrac{\Psi_1(x,y)}{Q/2\pi}\right] = K\cdot\dfrac{y}{a} \pm n\pi = \dfrac{\Psi_1(x_1,y_1)}{Q/2\pi} = 0$

* The asymptotes of the streamlines that pass through the Stagnation Point 1 are:

- $K\dfrac{y}{a} \pm n\pi = 0 \Rightarrow y = \begin{cases} -\pi a/K \\ 0 \\ +\pi a/K \end{cases}$

- Stagnation Point 2: $(R,\theta)_2 = (R_2, \pi) \Rightarrow (x,y)_2 = (-R_2 \cdot a ; 0)$ $<0$

  * Streamline that passes through the stagnation Point 2 ($\psi_2(x,y)$):

    - $$\frac{\psi_2(x_2, y_2)}{Q/2\pi} = atan\left(\frac{-1}{-R_2}\right) + atan\left(\frac{1}{-R_2}\right) = 0$$

  * When $x \to \pm\infty$:

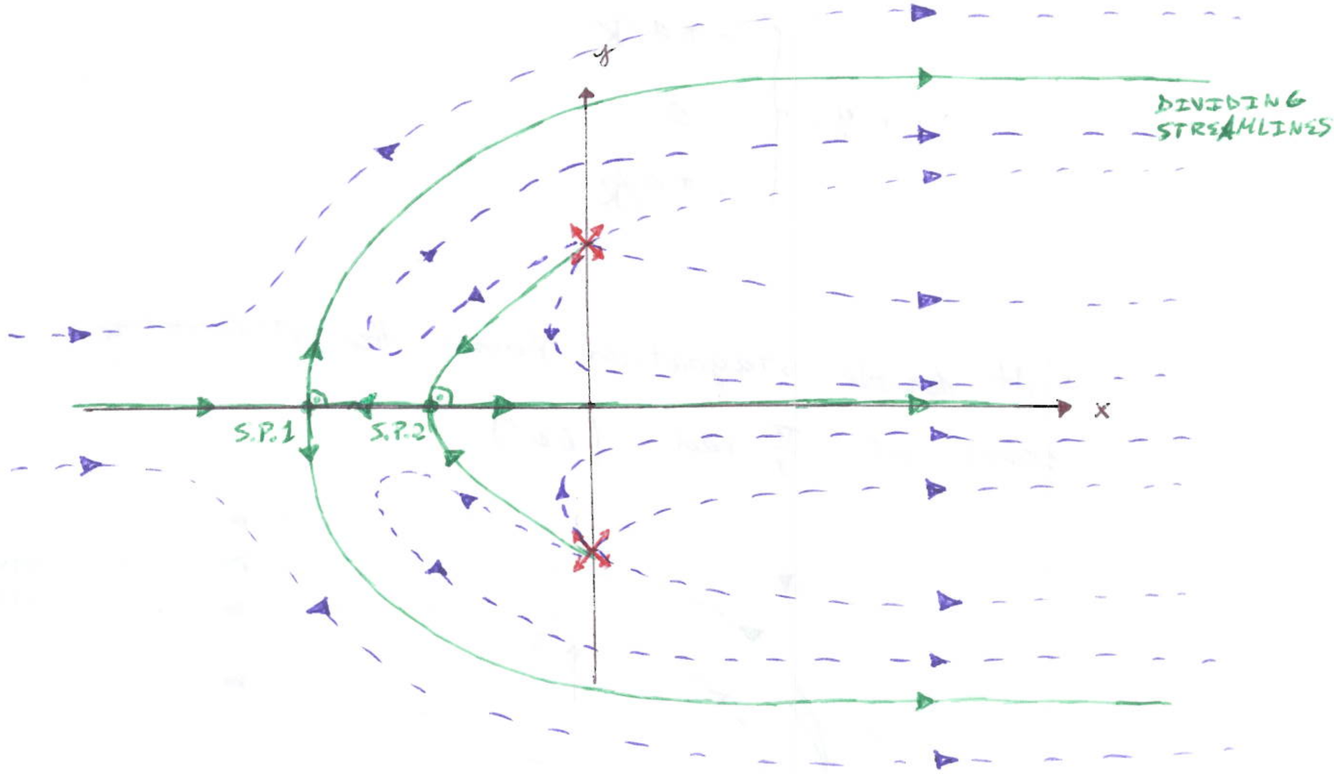
    - $$\lim_{x\to\pm\infty}\left[atan\left(\frac{\frac{y}{a}-1}{x/a}\right)\right] = \pm n \cdot \frac{\pi}{2}$$

    - $$\lim_{x\to\pm\infty}\left[\frac{\psi_2(x,y)}{Q/2\pi}\right] = K \cdot \frac{y}{a} \pm n\pi = \frac{\psi_2(x_2,y_2)}{Q/2\pi} = 0$$

  * The asymptotes of the streamlines that pass through the stagnation Point 2 are:

    - $$y = \begin{cases} -\pi a/K \\ 0 \\ +\pi a/K \end{cases}$$

- The streamlines only cross at singular points (sources, sinks, doublets, etc) and stagnation points. At simple stagnation points, they cross at $\frac{\pi}{2}$ rad (90°).




* $K = 1$:

- Double Stagnation Point: $(R, \theta)_1 = (1, \pi) \Rightarrow (x, y)_1 = (-a, 0)$

* Streamline that passes through the stagnation point ($\psi_1(x, y)$):

$$\frac{\psi_1(x_1, y_1)}{Q/2\pi} = atan\left(\frac{1}{a}\right) + atan\left(\frac{1}{-a}\right) = 0$$

* When $x \to \pm\infty$:

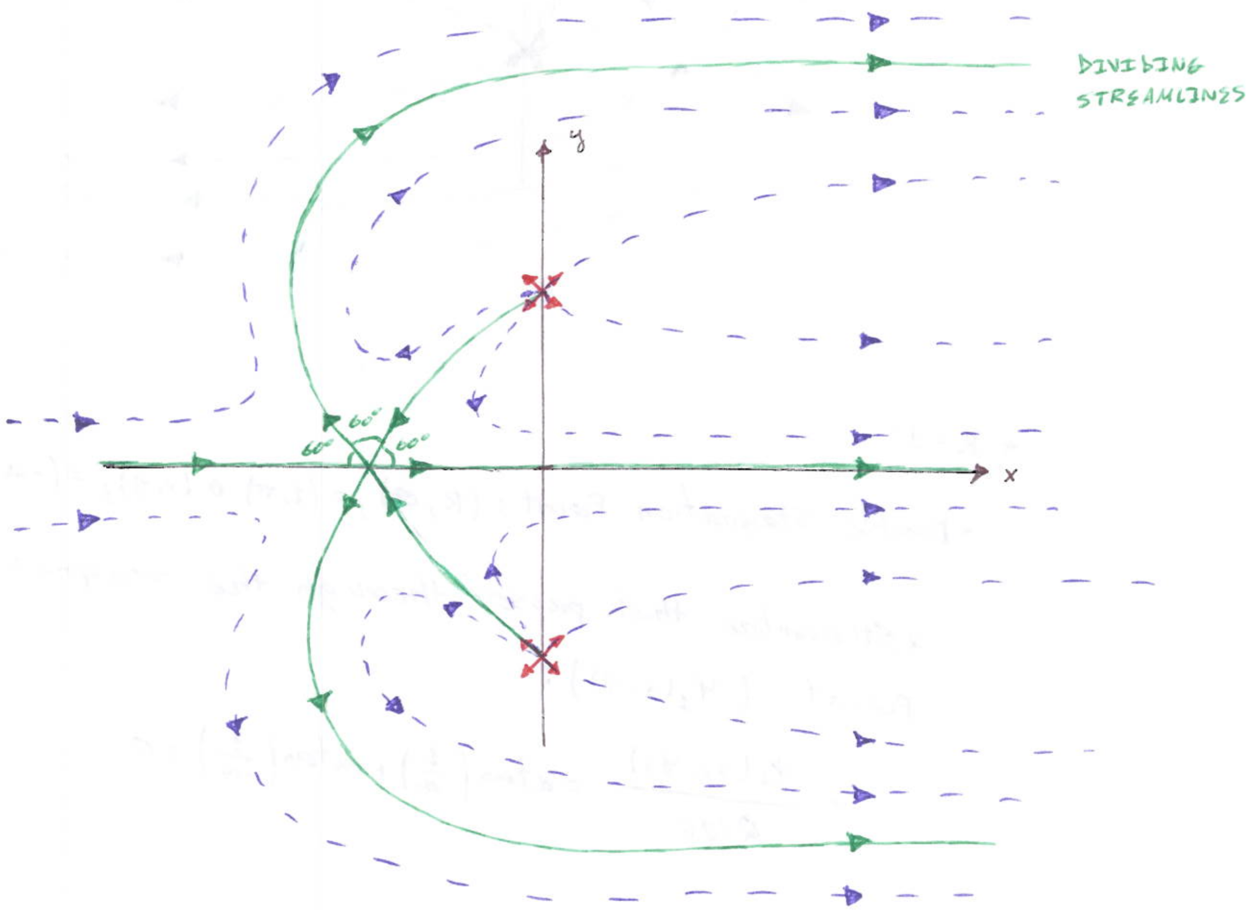
$$\lim_{x \to \pm\infty}\left[atan\left(\frac{\frac{y}{a} - 1}{x/a}\right)\right] = \pm n\frac{\pi}{2}$$

$$\lim_{x \to \pm\infty}\left[\frac{\psi_1(x, y)}{Q/2\pi}\right] = K\frac{y}{a} \pm n\pi = \frac{\psi_1(x_1, y_1)}{Q/2\pi} = 0$$

* The asymptotes of the streamlines that pass through the stagnation point are:

$$y = \begin{cases} -\pi a/K \\ 0 \\ +\pi a/K \end{cases}$$

- At double stagnation Points, the streamlines cross at $\frac{\pi}{3}$ rad $(60°)$.




* $K > 1$:

- Stagnation Point 1: $(R,\theta)_1 = \left[1, a\cos\left(\frac{-1}{K}\right)\right] \Rightarrow (x,y)_1 = \left(\frac{-a}{K}, a\sqrt{1-\frac{1}{K^2}}\right)$

  * streamline that passes through the stagnation Point 1 $(\psi_1(x,y))$:

  - $\frac{\psi_1(x_1,y_1)}{Q/2\pi} = \sqrt{K^2-1} + \left\{ a\tan\left(K - \sqrt{K^2-1}\right) + a\tan\left(-K - \sqrt{K^2-1}\right)\right\}$

* When $x \to \pm\infty$:

 - $$\lim_{x \to \pm\infty} \left[ \frac{\psi_1(x,y)}{Q/2\pi} \right] = K \frac{y}{a} \pm n\pi$$

* The asymptotes of the streamlines that pass through the stagnation Point 1 are:

 - $$y = \begin{cases} \psi_1 / a V_\infty \\ \dfrac{\psi_1}{a V_\infty} + \dfrac{\pi}{K} \end{cases}$$

- Stagnation Point 2: $(R,\theta)_2 = \left[1, -a\cos\left(\frac{-1}{K}\right)\right] \Rightarrow (x,y)_2 = \left(\frac{-a}{K}, -a\sqrt{1-\frac{1}{K^2}}\right)$

 * Streamline that passes through the stagnation Point 2 $(\psi_2(x,y))$:

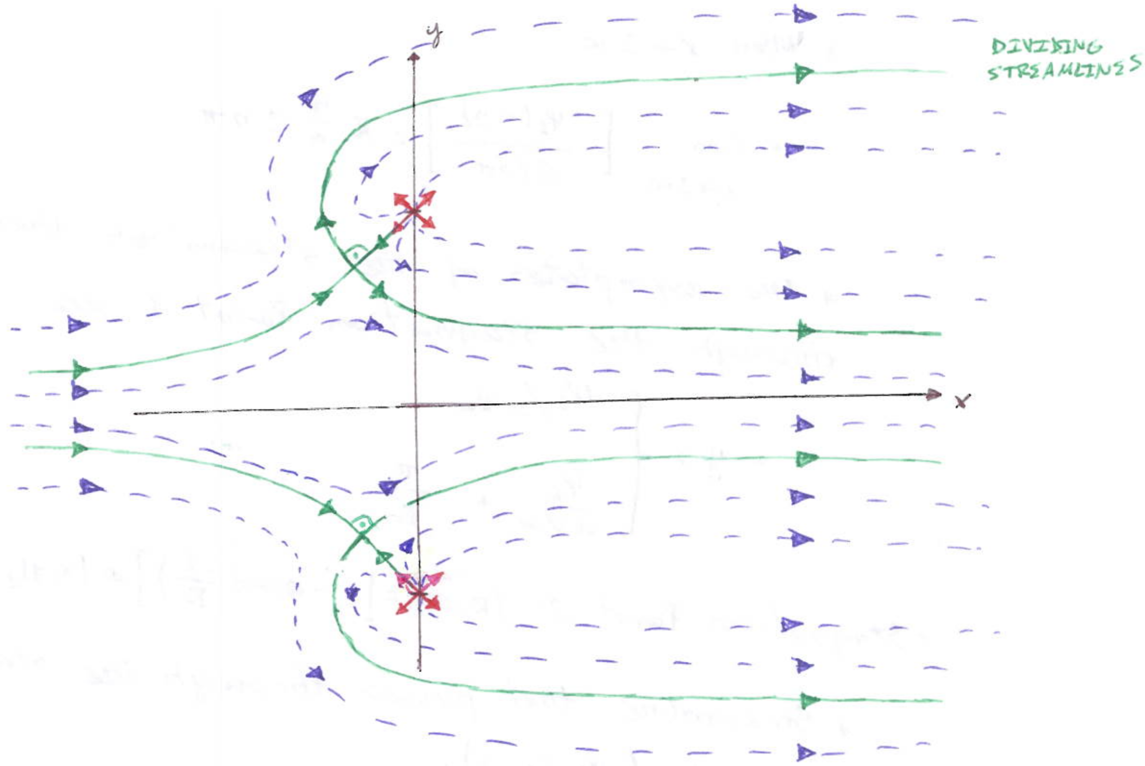
 - $$\frac{\psi_2(x_2,y_2)}{Q/2\pi} = -\sqrt{K^2-1} + \left\{ atan\left(K + \sqrt{K^2-1}\right) + atan\left(-K - \sqrt{K^2-1}\right) \right\}$$

 * When $x \to \pm\infty$:

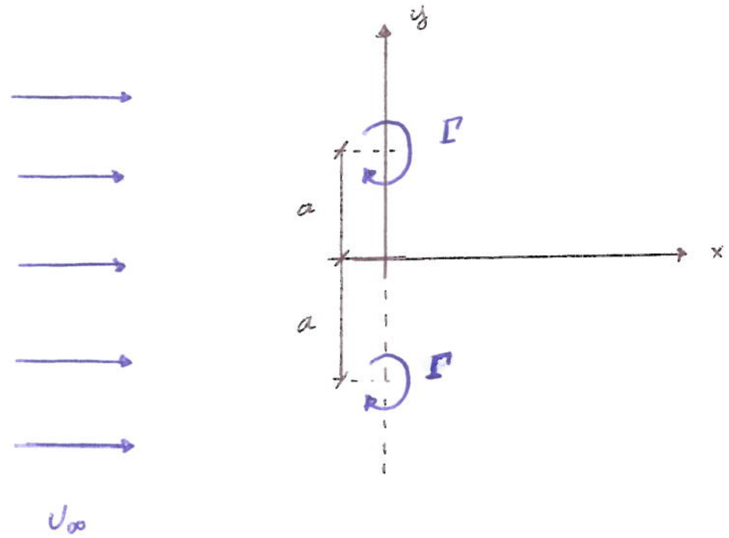
 - $$\lim_{x \to \pm\infty} \left[ \frac{\psi_2(x,y)}{Q/2\pi} \right] = K \frac{y}{a} \pm n\pi$$

 * The asymptotes of the streamlines that pass through the stagnation Point 2 are:

 - $$y = \begin{cases} \dfrac{\psi_2}{a V_\infty} \\ \dfrac{\psi_2}{a V_\infty} - \dfrac{\pi}{K} \end{cases}$$


y
x
DIVIDING STREAMLINES

Problem 1:




1)

⇒ stream function for a vertex at the origen in polar coordinates: $\psi_0 = \frac{\Gamma}{2\pi} \ln\left(\frac{r}{a}\right)$

⇒ In Cartesian coordinates: $\psi_0 = \frac{\Gamma}{2\pi} \ln\left[\frac{(x^2+y^2)^{1/2}}{a}\right] =$

$= \frac{\Gamma}{2\pi} \cdot \frac{1}{2} \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2\right]$

⇒ If the vertex is located at $(x, y) = (0, a)$:

$$\psi_1 = \frac{\Gamma}{2\pi} \cdot \frac{1}{2} \cdot \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} - 1\right)^2\right]$$

⇒ For a vertex located at $(x, y) = (0, -a)$:

$$\psi_2 = \frac{\Gamma}{2\pi} \cdot \frac{1}{2} \cdot \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} + 1\right)^2\right]$$

⇒ stream function of a uniform stream along the x-axis:

$$\psi_3 = U_\infty y$$

⇒ the stream function for the whole flow is the superposition of the three of them:

$$\boxed{\psi(x, y) = U_\infty y + \frac{\Gamma}{4\pi} \cdot \left\{ \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} + 1\right)^2\right] + \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} - 1\right)^2\right] \right\}}$$

2)

$$u = \frac{\partial \psi}{\partial y} = V_\infty + \frac{\Gamma}{4\pi}\left\{2 \cdot \frac{y+a}{x^2+(y+a)^2} + 2 \cdot \frac{y-a}{x^2+(y-a)^2}\right\} \Rightarrow$$

$$\Rightarrow \boxed{u(x,y) = V_\infty + \frac{\Gamma}{2\pi}\left\{\frac{y+a}{x^2+(y+a)^2} + \frac{y-a}{x^2+(y-a)^2}\right\}}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{-\Gamma}{4\pi} \cdot \left\{\frac{2x}{x^2+(y+a)^2} + \frac{2x}{x^2+(y-a)^2}\right\} \Rightarrow$$

$$\Rightarrow \boxed{v(x,y) = \frac{-\Gamma}{2\pi}\left\{\frac{x}{x^2+(y+a)^2} + \frac{x}{x^2+(y-a)^2}\right\}}$$

3)

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \Rightarrow \phi = \int u\, dx = \int\left\{V_\infty + \frac{\Gamma}{2\pi}\left[\frac{y+a}{x^2+(y+a)^2} + \frac{y-a}{x^2+(y-a)^2}\right]\right\}dx =$$

$$= V_\infty x + \frac{\Gamma}{2\pi}\left\{\arctan\left(\frac{x}{y+a}\right) + \arctan\left(\frac{x}{y-a}\right)\right\} + F(y)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\Gamma}{2\pi}\left\{\frac{x}{x^2+(y+a)^2} + \frac{x}{x^2+(y-a)^2}\right\} = \frac{\partial \phi}{\partial y} =$$

$$= \frac{\Gamma}{2\pi} \cdot \left\{\frac{-x}{x^2+(y+a)^2} - \frac{x}{x^2+(y-a)^2}\right\} + F'(y) \Rightarrow F'(y) = 0 \Rightarrow$$

$$\Rightarrow F(y) = cte.$$

$$\boxed{\phi(x,y) = V_\infty \cdot x + \frac{\Gamma}{2\pi}\left\{\operatorname{atan}\left(\frac{x}{y+a}\right) + \operatorname{atan}\left(\frac{x}{y-a}\right)\right\}}$$

4)

$$\frac{u}{\Gamma/2\pi a} = \frac{U_\infty}{\Gamma/2\pi a} + \left\{ \frac{\frac{y}{a}+1}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2} + \frac{\frac{y}{a}-1}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2} \right\} =$$

$$= K + \frac{R\sin\theta+1}{(R\cos\theta)^2+(R\sin\theta+1)^2} + \frac{R\sin\theta-1}{(R\cos\theta)^2+(R\sin\theta-1)^2} =$$

$$= K + \frac{R\sin\theta+1}{R^2+2R\sin\theta+1} + \frac{R\sin\theta-1}{R^2-2R\sin\theta+1} \Rightarrow$$

$$\Rightarrow \boxed{\frac{u}{\Gamma/2\pi a} = K + \frac{2R\sin\theta\,(R^2-1)}{(R^2+1)^2-4R^2\sin^2\theta}}$$

$$\frac{v}{\Gamma/2\pi a} = \frac{-\frac{x}{a}}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2} - \frac{x/a}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2} =$$

$$= \frac{-R\cos\theta}{(R\cos\theta)^2+(R\sin\theta+1)^2} - \frac{R\cos\theta}{(R\cos\theta)^2+(R\sin\theta-1)^2} =$$

$$= -R\cdot\cos\theta\cdot\left\{\frac{1}{R^2+2R\sin\theta+1} + \frac{1}{R^2-2R\sin\theta+1}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\frac{v}{\Gamma/2\pi a} = \frac{-2R\cos\theta\,(R^2+1)}{(R^2+1)^2-4R^2\sin\theta}}$$

5)

$$\frac{v}{\Gamma/2\pi a} = \frac{-2R\cos\theta\,(R^2+1)}{(R^2+1)^2-4R^2\sin\theta} = 0 \Rightarrow \begin{cases} R^2 = -1 \Rightarrow \text{NO!} \\ \boxed{R = 0} \\ \cos\theta = 0 \Rightarrow \boxed{\theta = \pi/2},\ \boxed{\theta = 3\pi/2} \end{cases}$$

6)

$$\frac{u(R=0)}{\Gamma/2\pi a} = K = 0$$ → No solution unless $U_\infty = 0$

$\sin(\frac{\pi}{2}) = 1$

$$\frac{u(\theta=\frac{\pi}{2})}{\Gamma/2\pi a} = K + \frac{2R(R^2-1)}{(R^2+1)^2 - 4R^2} = K + \frac{2R(R^2-1)}{R^4+2R^2+1-4R^2} =$$

$$= K + \frac{2R\cancel{(R^2-1)}}{(R^2-1)^{\cancel{2}}} = K + \frac{2R}{R^2-1} = 0 \Rightarrow$$

$$\Rightarrow KR^2 - K + 2R = 0 \Rightarrow R = \frac{-2 \pm \sqrt{4+4K^2}}{2K} \Rightarrow$$

$$\Rightarrow \quad R = -\frac{1}{K} \pm \sqrt{1+\frac{1}{K^2}}$$

R must be greater than 0 $\Rightarrow$ $R = -\frac{1}{K} + \sqrt{1+\frac{1}{K^2}}$

$\sin(\frac{3\pi}{2}) = -1$

$$u(\theta=\frac{3\pi}{2}) = K + \frac{2R(-1)(R^2-1)}{(R^2+1)^2-4R^2} = K - \frac{2R}{R^2-1} = 0 \Rightarrow$$

$$\Rightarrow KR^2 - 2R - K = 0 \Rightarrow R = \frac{2 \pm \sqrt{4+4K^2}}{2K} \Rightarrow$$

$$\Rightarrow R = \frac{1}{K} \pm \sqrt{1+\frac{1}{K^2}} \Rightarrow R = \frac{1}{K} + \sqrt{1+\frac{1}{K^2}}$$

So the two stagnation points are:

$$\boxed{(R,\theta)_1 = \left[-\frac{1}{K} + \sqrt{1+\frac{1}{K^2}}, \frac{\pi}{2}\right]}$$

$$\boxed{(R,\theta)_2 = \left[\frac{1}{K} + \sqrt{1+\frac{1}{K^2}}, \frac{3\pi}{2}\right]}$$

In Cartesian coordinates:

$$\boxed{(x_1, y_1) = \left(0, -\frac{1}{K} + \sqrt{1+\frac{1}{K^2}}\right)\cdot a}$$

$$\boxed{(x_2, y_2) = \left(0, -\frac{1}{K} - \sqrt{1+\frac{1}{K^2}}\right)\cdot a}$$

7) A streamline that passes through a stagnation point is a dividing streamline.

The non-dimensional stream function is:

$$\frac{\psi(x,y)}{\Gamma/2\pi} = K\frac{y}{a} + \frac{1}{2}\left\{ \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}+1\right)^2\right] + \ln\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}-1\right)^2\right]\right\}$$

The value of the stream function for the streamline that passes by the first stagnation point is:

$$\frac{\psi_1(x_1,y_1)}{\Gamma/2\pi} = K\frac{y_1}{a} + \frac{1}{2}\left\{ \ln\left[\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{a}+1\right)^2\right] + \ln\left[\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{a}-1\right)^2\right]\right\} =$$

$$= K\frac{y_1}{a} + \frac{1}{2}\ln\left\{\left(\frac{y_1}{a}+1\right)^2\left(\frac{y_1}{a}-1\right)^2\right\} =$$

$$= K\frac{y_1}{a} + \frac{1}{2}\ln\left\{\left[\left(\frac{y_1}{a}\right)^2 - 1\right]^2\right\} =$$

$$= -1 + \sqrt{K^2+1} + \frac{1}{2}\ln\left\{\left[\left(-\frac{1}{K} + \sqrt{1+\frac{1}{K^2}}\right)^2 - 1\right]^2\right\} =$$

$$= -1 + \sqrt{K^2+1} + \ln\left\{\frac{2}{K^2} - \frac{2}{K^2}\sqrt{K^2+1}\right\}$$

and for the second:

$$\frac{\psi_2(x_2,y_2)}{\Gamma/2\pi} = K\frac{y_2}{a} + \ln\left[\left(\frac{y_2}{a}\right)^2 - 1\right] =$$

$$= -1 - \sqrt{K^2+1} + \ln\left\{\frac{2}{K^2} + \frac{2}{K^2}\sqrt{K^2+1}\right\}$$

Assume that, for the dividing streamlines, $x \gg y$ when $x \to \pm\infty$:

So:

$$\frac{\psi_1}{\Gamma/2\pi} = K\frac{y_1}{a} + \ln\left[\left(\frac{y_1}{a}\right)^2 - 1\right] = \frac{y}{a} + \frac{1}{2}\left\{\ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2\right] + \ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2\right]\right\} \sim$$

$$\underset{x\to\pm\infty}{\sim} K\frac{y}{a} + \frac{1}{2}\left[\ln\left(\frac{x}{a}\right)^2 + \ln\left(\frac{x}{a}\right)^2\right] =$$

$$= K\frac{y}{a} + 2\cdot\ln\left(\frac{x}{a}\right) \Rightarrow$$

$$\Rightarrow K\left(\frac{y}{a} - \frac{y_1}{a}\right) = -\ln\left[\frac{(x/a)^2}{\left(\frac{y_1}{a}\right)^2 - 1}\right] \Rightarrow$$

$$\Rightarrow \frac{y}{a} = \frac{y_1}{a} - \frac{1}{K}\ln\left[\frac{(x/a)^2}{\left(\frac{y_1}{a}\right)^2 - 1}\right] = \left\{\frac{y_1}{a} + \frac{1}{K}\ln\left[\left(\frac{y_1}{a}\right)^2 - 1\right]\right\} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2$$

$$\frac{\psi_2}{\Gamma/2\pi} = K\frac{y_2}{a} + \ln\left[\left(\frac{y_2}{a}\right)^2 - 1\right] = K\frac{y}{a} + \frac{1}{2}\left\{\ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2\right] + \ln\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2\right]\right\} \sim$$

$$\underset{x\to\pm\infty}{\sim} K\frac{y}{a} + \frac{1}{2}\left[\ln\left(\frac{x}{a}\right)^2 + \ln\left(\frac{x}{a}\right)^2\right] = K\frac{y}{a} + \ln\left(\frac{x}{a}\right)^2 \Rightarrow$$

$$\Rightarrow K\left(\frac{y}{a} - \frac{y_2}{a}\right) = -\ln\left[\frac{(x/a)^2}{\left(\frac{y_2}{a}\right)^2 - 1}\right] \Rightarrow$$

$$\Rightarrow \frac{y}{a} = \frac{y_2}{a} - \frac{1}{K}\ln\left[\frac{(x/a)^2}{\left(\frac{y_2}{a}\right)^2 - 1}\right] = \left\{\frac{y_2}{a} + \frac{1}{K}\ln\left[\left(\frac{y_2}{a}\right)^2 - 1\right]\right\} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2$$

So the dividing streamlines are given by:

$$\frac{y}{a} = \frac{y_1}{a} - \frac{1}{2K}\left\{\ln\left[\frac{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2}{\left(\frac{y_1}{a}\right)^2-1}\right] + \ln\left[\frac{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2}{\left(\frac{y_1}{a}\right)^2-1}\right]\right\}$$

$$\frac{y}{a} = \frac{y_2}{a} - \frac{1}{2K}\left\{\ln\left[\frac{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}+1\right)^2}{\left(\frac{y_2}{a}\right)^2-1}\right] + \ln\left[\frac{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}-1\right)^2}{\left(\frac{y_2}{a}\right)^2-1}\right]\right\}$$

and their asymptotes for big $x$ are:

$$\frac{y}{a} \sim \left\{\frac{y_1}{a} + \frac{1}{K}\ln\left[\left(\frac{y_1}{a}\right)^2 - 1\right]\right\} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2 = y_{c1} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2$$

$$\frac{y}{a} \sim \left\{\frac{y_2}{a} + \frac{1}{K}\ln\left[\left(\frac{y_2}{a}\right)^2 - 1\right]\right\} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2 = y_{c2} - \frac{1}{K}\ln\left(\frac{x}{a}\right)^2$$

There are several cases depending on $K$.

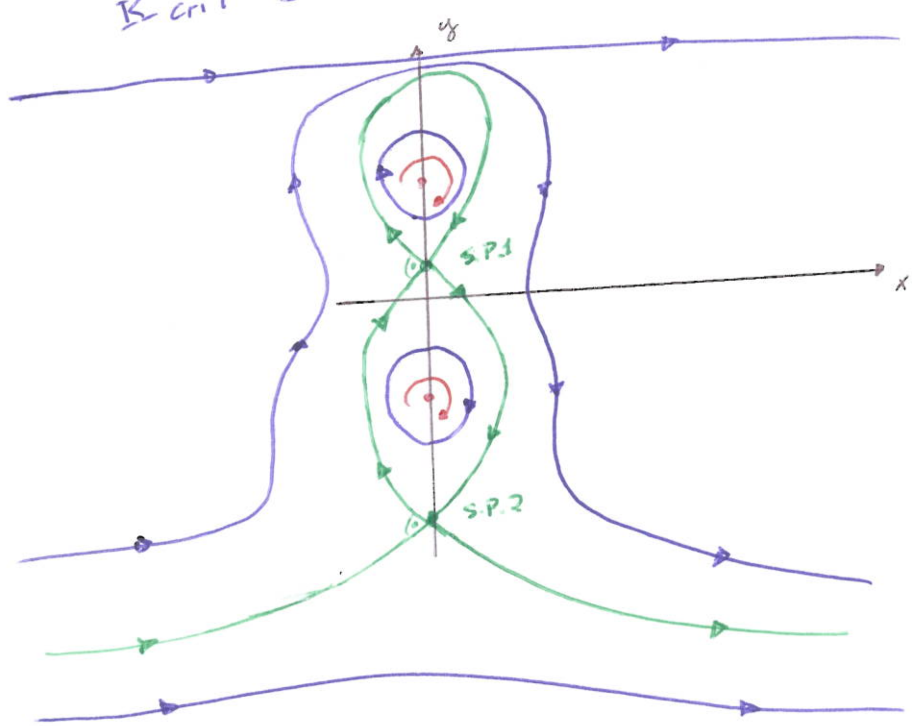
⇨ First, there is a critical case where both dividing streamlines coincide:

$$\frac{\psi_1}{\Gamma/2\pi} = \frac{\psi_2}{\Gamma/2\pi} \Rightarrow K\left(\frac{y_1}{a} - \frac{y_2}{a}\right) = -\ln\left[\frac{\left(\frac{y_1}{a}\right)^2 - 1}{\left(\frac{y_2}{a}\right)^2 - 1}\right] \Rightarrow$$

$$\Rightarrow \ln\left\{\frac{\frac{1}{K} - \sqrt{1 + \frac{1}{K^2}}}{\frac{1}{K} + \sqrt{1 + \frac{1}{K^2}}}\right\} + 2K\sqrt{1 + \frac{1}{K^2}} = 0$$

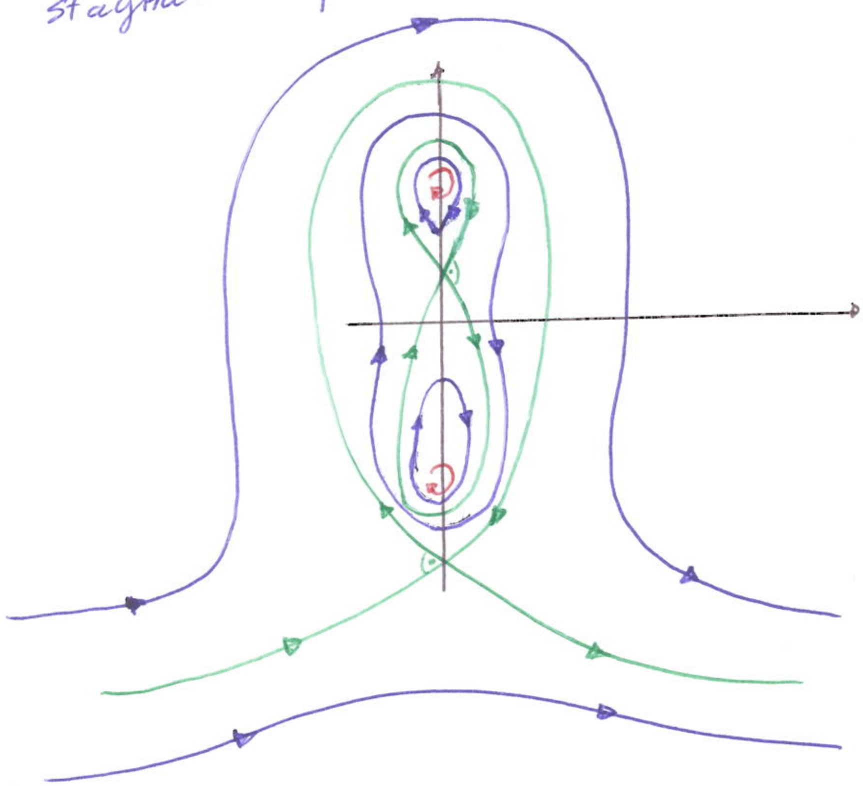
Solving numerically this equations I get:

$$K_{crit} \simeq 0.6628$$



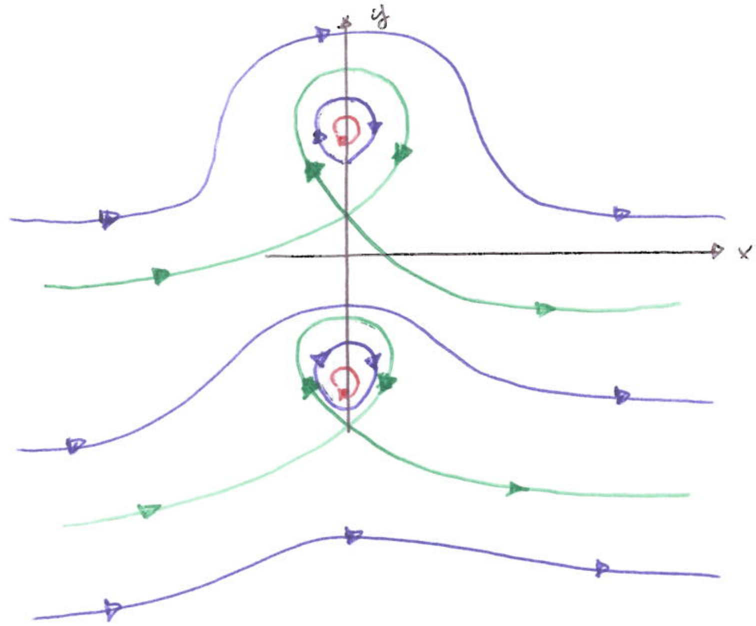
⇨ Second, $K < K_{crit}$, i.e., the vortices are "stronger" than the horizontal stream:

can calculate numerically that the stream line passing through the stagnation point 1 cuts three times the y-axis, and the one passing through the stagnation point 2, cuts only 2:



⇨ Third, $K > K_{crit}$, i.e., the vortices are "weaker" than the horizontal stream.

can calculate numerically that the stream line passing through the stagnation point 1 cuts two times the y-axis, and the one passing throug the stagnation point 2, also 2 times:




⇨ There is a subcase in this one. when $K > K_{crit}$ and $y_{c1} > y_{c2}$, we have more solutions. This means that the stream lines pass more than twice throug the y-axis.