Solution to Homework 3

February 3, 2017

Problem 1

- Increased cloud cover that from increased water vapor in atmosphere will increase the albedo of the atmosphere, α_1, reducing the transmitted sunlight flux, leading to lower absorbed power and temperature of the atmosphere.
- A decrease in sea ice coverage reduces the albedo of the Earth's surface, α_2, and thus leads to higher absorbed power of the surface.
- The increase in methane gas emission reduces the IR transmission coefficient, β_{IR}. Therefore, less power is transmitted out to the space and more power is absorbed by the atmosphere. It gives rise to a heating of the atmosphere and an increase in the rate of IR radiation transfer between the surface and the atmosphere. As a result, the overall temperature of the surface and atmosphere will increase.

Problem 2

The total atmospheric mass is roughly $5 \times 10^{18}\,\mathrm{kg}$.

- The current concentration of $\mathrm{CO_2}$ is about 400 ppmv. The current mass of $\mathrm{CO_2}$ is $m_{\mathrm{CO_2}} = \frac{44}{29} \times 400 \times 10^{-6} \times 5 \times 10^{18}\,\mathrm{kg} = 3.0 \times 10^{15}\,\mathrm{kg}$, and the mass of carbon is $m_{\mathrm{C}} = \frac{12}{44} m_{\mathrm{CO_2}} = 8.3 \times 10^{14}\,\mathrm{kg}$.
- In preindustrial era the concentration of $\mathrm{CO_2}$ is about 280 ppmv. The corresponding mass of $\mathrm{CO_2}$ is $m_{\mathrm{CO_2}} = \frac{44}{29} \times 280 \times 10^{-6} \times 5 \times 10^{18}\,\mathrm{kg} = 2.1 \times 10^{15}\,\mathrm{kg}$, and the mass of carbon is $m_{\mathrm{C}} = \frac{12}{44} m_{\mathrm{CO_2}} = 5.8 \times 10^{14}\,\mathrm{kg}$.
- The increase in C content in the atmosphere is about $8.3 \times 10^{14} - 5.8 \times 10^{14} = 2.5 \times 10^{14}\,\mathrm{kg}$.
- The total mass of C injected into the atmosphere from fossil fuels since mid 19th century is about $3.9 \times 10^{14}\,\mathrm{kg}$.[1]
- The amount of injected C mass is about 56% greater than the change in atmospheric C mass during the same time. It indicates that human activities, particularly the burning of fossil fuels since the Industrial Revolution, significantly increase the $\mathrm{CO_2}$ emission and C content in the atmosphere. The increases and current atmospheric levels are the result of the competition between sources (the

[1] Reference see http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html

emissions from human activities and natural systems) and sinks (their removal from the atmosphere by conversion to different chemical compounds–for example, CO2 is removed by photosynthesis and conversion to carbonates).

Problem 3

a)

The carbon balance model can then be written as

$$\frac{\partial}{\partial t}\delta M_C(t) = Q_C(t) - \frac{\delta M_C(t)}{t_{\rm net}},$$

where the carbon injection rate $Q_C(t) = 8$ Gt/yr for $t \geq 0$ and $t_{\rm net} = 100$ yr. The deviation of the atmospheric C inventory from equilibrium is then given as

$$\delta M_C(t) = Q_C t_{\rm net}\left(1 - e^{-t/t_{\rm net}}\right) = 800\left(1 - e^{-t/100}\right).$$

The mass of air in atmosphere is about 5×10^6 Gt. The future CO_2 concentration (ppmv) is then written as

$$\rho_{CO_2}(t) = 400 + \frac{29}{12} \times \frac{1 \times 10^6}{5 \times 10^6} \times 800\left(1 - e^{-t/100}\right) = \frac{40}{3}\left(59 - 29e^{-\frac{t}{100}}\right).$$

The result is plotted in Fig. 1.

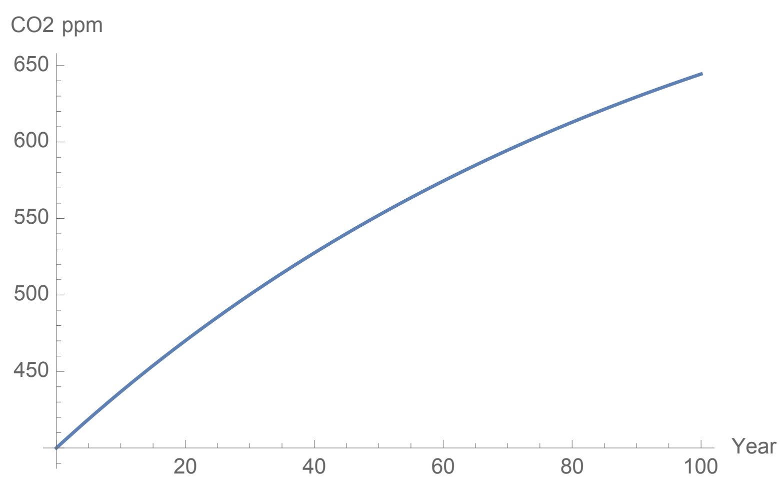

Figure 1: The evolution of CO_2 concentration with constant carbon source.

b)

Since 1 ppmv of CO_2 gives about 2 Gt of carbon and current CO_2 is about 400 ppmv, the current mass of carbon in the atmosphere is $M_C(0) \approx 800$ Gt. The evolution of the IR transmission coefficient in time is then written as

$$\begin{aligned}\beta(t) &= \beta_0 \exp\left(-\frac{\sigma_{\rm gg} d}{V}\delta M_C(t)\right),\\ &= \beta_0 \exp\left(\frac{\ln\beta_0}{M_C(0)}\delta M_C(t)\right),\\ &= 0.05\exp\left[3\left(e^{-t/100}-1\right)\right].\end{aligned}$$

The result is shown in Fig. 2.

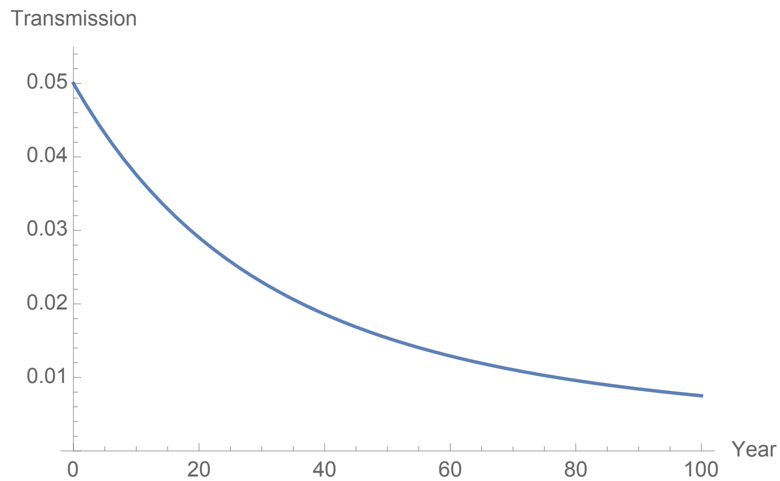

Figure 2: The evolution of IR transmission coefficient in time.

c)

Solve the equations from Problem 3 of HW 2 using new parameters,

$$\begin{aligned}E_s &= 0.6E_a + (1-\alpha_2)\beta_{\rm vis} I,\\ E_a &= \left(1-\beta_{\rm IR}\right)E_s + A_1,\\ A_1 &= \left[1-\alpha_1-\beta_{\rm vis}+\beta_{\rm vis}\alpha_2\left(1-\beta_{\rm vis}\right)\right]I.\end{aligned}$$

The results for E_s and E_a are,

$$\begin{aligned}E_s &= \frac{334}{\beta_{\rm IR}+0.67},\\ E_a &= -\frac{275(1.4-\beta_{\rm IR})}{\beta_{\rm IR}+0.67}.\end{aligned}$$

The corresponding temperatures are,

$$T_s = 277\left(\frac{1}{\beta_{\rm IR}+0.67}\right)^{0.25} = 277\left(\frac{1}{0.67+0.05e^{-3\left(1-e^{-\frac{t}{100}}\right)}}\right)^{0.25},$$

$$T_a = 264\left(\frac{1.4-\beta_{\rm IR}}{\beta_{\rm IR}+0.67}\right)^{0.25} = 264\left(\frac{1.4-0.05e^{-3\left(1-e^{-\frac{t}{100}}\right)}}{0.67+0.05e^{-3\left(1-e^{-\frac{t}{100}}\right)}}\right)^{0.25}.$$

The results are plotted in Fig. 3.

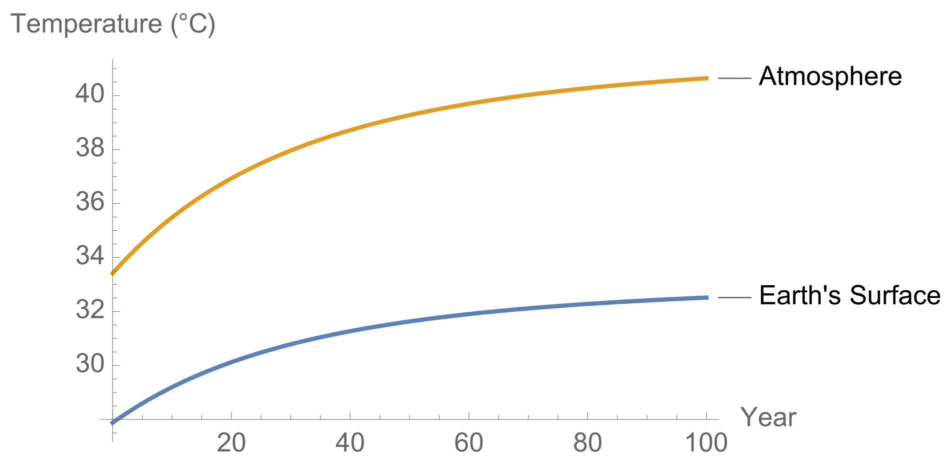

Figure 3: The evolution of temperatures of the Earth's surface and the atmosphere.

d)

If the carbon source is not constant but growing 3% per year, i.e. $Q_C(t) = 8(1+0.03)^t$ Gt/yr. The carbon balance equation becomes,

$$\frac{\partial}{\partial t}\delta M_C(t) + \frac{\delta M_C(t)}{t_{\rm net}} = 1.03^t Q_C(0).$$

Using integrating factors method, we can obtain the general solution to this equation,

$$\begin{aligned}\delta M_C(t) &= e^{-t/t_{\rm net}}\int_0^t 1.03^\tau Q_C(0)e^{\tau/t_{\rm net}}\,d\tau + \delta M_C(0)e^{-t/t_{\rm net}},\\ &= 8e^{-t/100}\int_0^t 1.03^\tau e^{\tau/100}\,d\tau,\\ &\approx 200e^{-\frac{t}{100}}\left(e^{0.04t}-1\right),\end{aligned}$$

where the last term on RHS of first line is neglected due to the zero initial value. The evolution of CO_2 concentration (ppm) is then written as

$$\rho_{CO_2}(t) = \frac{290}{3}e^{-\frac{t}{100}}\left(e^{0.04t}-1\right)+400$$

The result is plotted in 4.

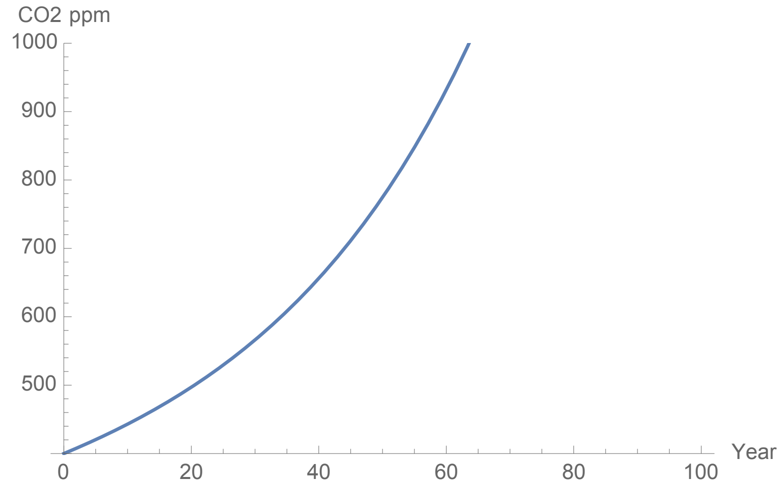

Figure 4: The evolution of CO_2 concentration with growing carbon source.

The evolution of the IR transmission coefficient can be expressed as

$$\begin{aligned}\beta(t) &= \beta_0 \exp\left(-\frac{\sigma_{gg} d}{V}\delta M_C(t)\right),\\ &= \beta_0 \exp\left(\frac{\ln\beta_0}{M_C(0)}\delta M_C(t)\right),\\ &= 0.05 \exp\left[-\frac{3}{4}e^{-\frac{t}{100}}\left(e^{0.04t}-1\right)\right].\end{aligned}$$

The result is plotted in Fig. 5

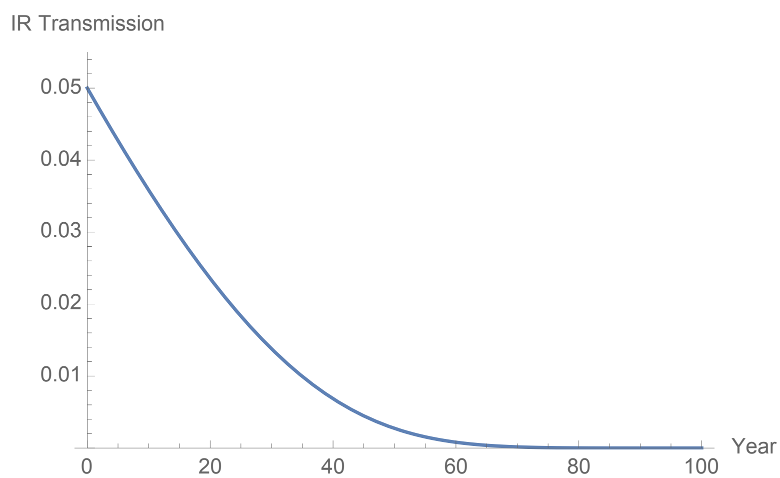

Figure 5: The evolution of the IR transmission coefficient in time.

Then substitute the $\beta(t)$ into the results we got in c), we can obtain the expressions for temperatures,

$$\begin{aligned}T_s &= 277\left(\frac{1}{0.67+0.05e^{-\frac{3}{4}e^{-0.01t}(e^{0.04t}-1)}}\right)^{0.25},\\ T_a &= 264\left(\frac{1.4-0.05e^{-\frac{3}{4}e^{-0.01t}(e^{0.04t}-1)}}{0.67+0.05e^{-\frac{3}{4}e^{-0.01t}(e^{0.04t}-1)}}\right)^{0.25}.\end{aligned}$$

Results are plotted in Fig. 6

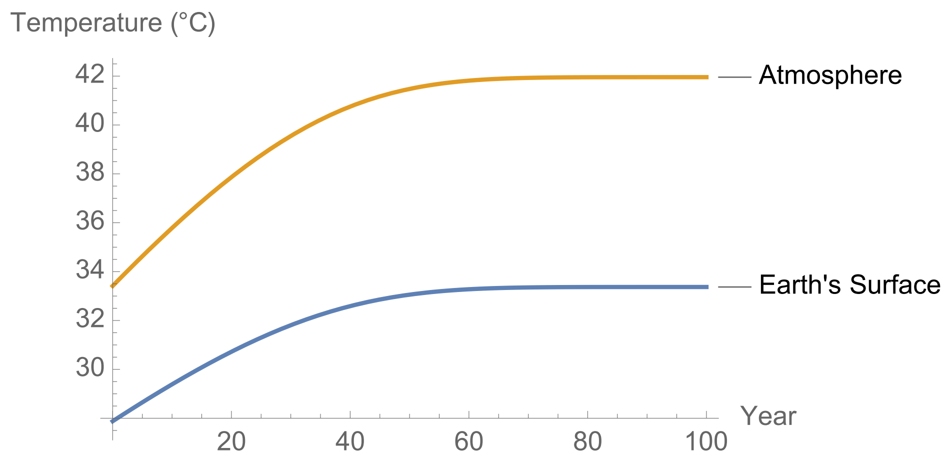

Figure 6: The evolution of the temperatures of Earth's surface and the atmosphere.

e)

For a short time compare to the t_{net} ($t \ll t_{\text{net}}$), the simple model we employed gives roughly the same trend of atmospheric CO_2 concentration as IPCC result. The rise in temperature estimated in our model, due to the increase in carbon content in atmosphere, is larger than published results.

Problem 4

The carbon balance model becomes

$$\frac{\partial}{\partial t}\delta M_{\text{CO}_2}(t) = Q_{\text{CO}_2}(t) - \frac{\delta M_{\text{CO}_2}(t)}{t_{\text{net}}},$$

where the CO_2 injection rate is $Q_{\text{CO}_2} = 100\,\delta(t)$ Gt/yr and the effective residence time of CO_2 is $t_{\text{net}} = 100$ yr. The solution is written as

$$\begin{aligned}\delta M_{\text{CO}_2}(t) &= e^{-t/t_{\text{net}}}\int_0^t 100\,\delta(\tau)e^{\tau/t_{\text{net}}}\,d\tau,\\ &= 100\exp(-t/100).\end{aligned}$$

The corresponding concentration of CO_2 is then

$$\begin{aligned}\rho_{\text{CO}_2}(t) &= 400 + \frac{29}{44}\times\frac{1\times10^6}{5\times10^6}\times 100\exp(-t/t_{\text{net}}),\\ &= 400 + \frac{145}{11}\exp(-t/100).\end{aligned}$$

The evolution of CO_2 concentration is plotted in Fig. 7. *Remark*: CO_2 concentration jumps from 400 to about 413 at $t = 0$, due to the instantaneous injection of carbon, and then decays exponentially.

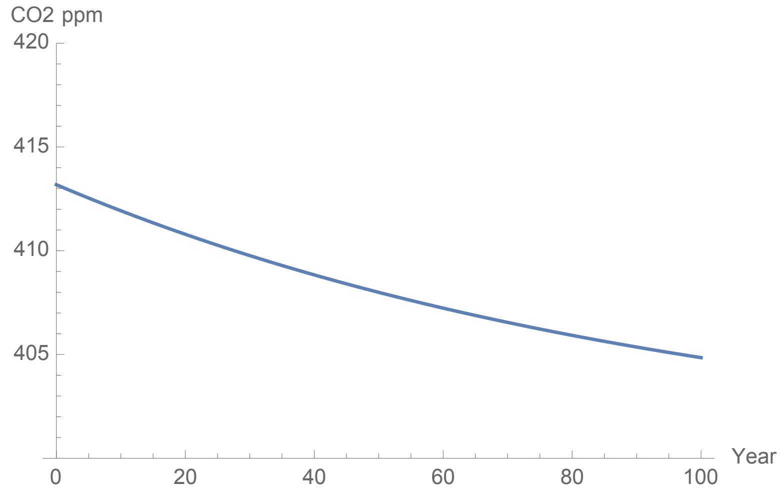

Figure 7: The evolution of CO_2 concentration with instantaneously injected 100 Gt of CO_2.

The evolution of the IR transmission coefficient can be expressed as

$$\begin{aligned}\beta(t) &= \beta_0 \exp\left(-\frac{\sigma_{\mathrm{gg}} d}{V} \delta M_C(t)\right), \\ &= \beta_0 \exp\left(\frac{\ln \beta_0}{M_C(0)} \delta M_{\mathrm{CO}_2}(t) \frac{12}{44}\right), \\ &= 0.05 \exp\left[-\frac{9}{88} e^{-\frac{t}{100}}\right].\end{aligned}$$

The evolution of IR transmission coefficient is plotted in Fig. 8. *Remark*: β_{IR} jumps from 0.05 to about 0.45 at $t = 0$ and then recovers.

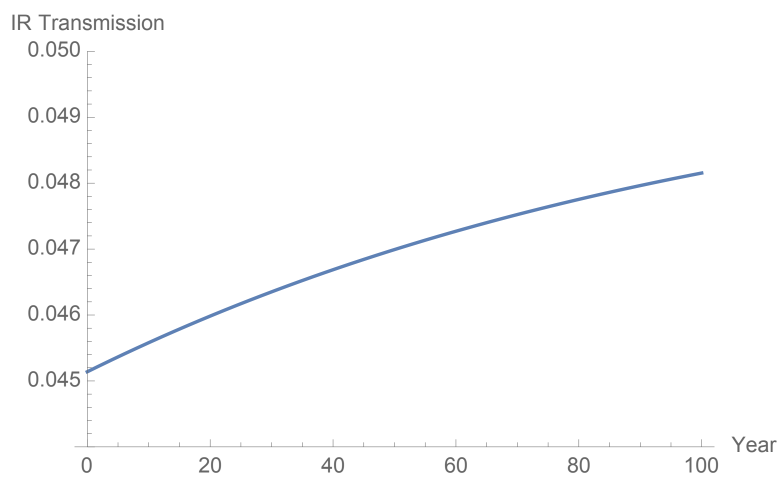

Figure 8: The evolution of IR transmission coefficient with instantaneously injected 100 Gt of CO_2.

The resulting temperature would become

$$T_s = 277\left(\frac{1}{0.67 + 0.05e^{-\frac{9}{88}e^{-0.01t}}}\right)^{0.25},$$
$$T_a = 264\left(\frac{1.4 - 0.05e^{-\frac{9}{88}e^{-0.01t}}}{0.67 + 0.05e^{-\frac{9}{88}e^{-0.01t}}}\right)^{0.25}.$$

The evolution of temperatures of the Earth's surface and the atmosphere are plotted in Fig. 9. *Remark*: T_s and T_a have small jumps at $t = 0$ and then recover to original values.

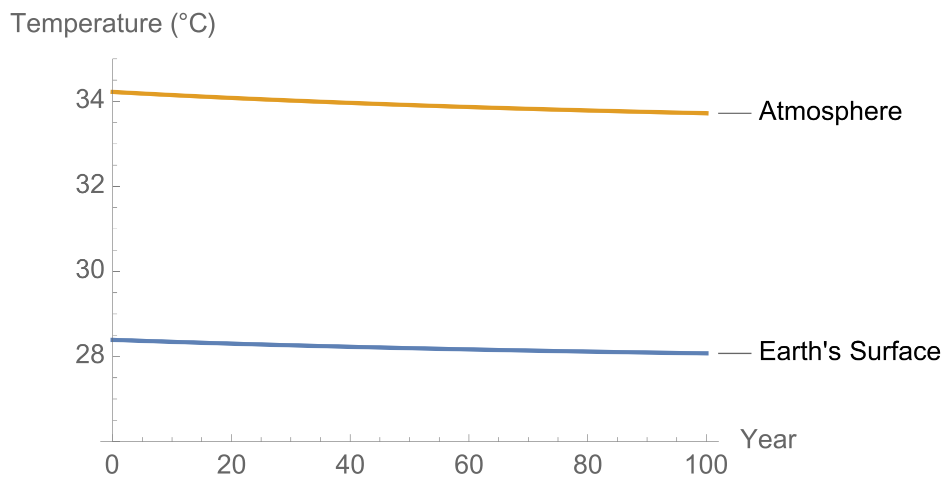

Figure 9: The evolution of temperatures of the Earth's surface and the atmosphere with instantaneously injected 100 Gt of carbon.

Problem 5

If the CO_2 source is 40 Gt/yr and growing 3% per year, i.e. $Q_{CO_2}(t) = 40(1+0.03)^t$ Gt/yr. The carbon balance equation becomes,

$$\frac{\partial}{\partial t}\delta M_{CO_2}(t) + \frac{\delta M_{CO_2}(t)}{t_{\text{net}}} = 1.03^t Q_{CO_2}(0).$$

Using integrating factors method, we can obtain the general solution to this equation,

$$\begin{aligned}\delta M_{CO_2}(t) &= e^{-t/t_{\text{net}}}\int_0^t 1.03^\tau Q_{CO_2}(0)e^{\tau/t_{\text{net}}}\,d\tau + \delta M_{CO_2}(0)e^{-t/t_{\text{net}}},\\ &= 40\,e^{-t/100}\int_0^t 1.03^\tau e^{\tau/100}\,d\tau,\\ &\approx 1000\,e^{-\frac{t}{100}}\left(e^{0.04t} - 1\right),\end{aligned}$$

Evolution of the IR transmission coefficient is

$$
\begin{aligned}
\beta(t) &= \beta_0 \exp\left(-\frac{\sigma_{gg} d}{V}\delta M_C(t)\right), \\
&= \beta_0 \exp\left(\frac{\ln \beta_0}{M_C(0)}\delta M_{CO_2}(t)\frac{12}{44}\right), \\
&= 0.05 \exp\left[-\frac{45}{44}e^{-t/100}(e^{0.04t}-1)\right].
\end{aligned}
$$

The result is plotted in Fig. 10.

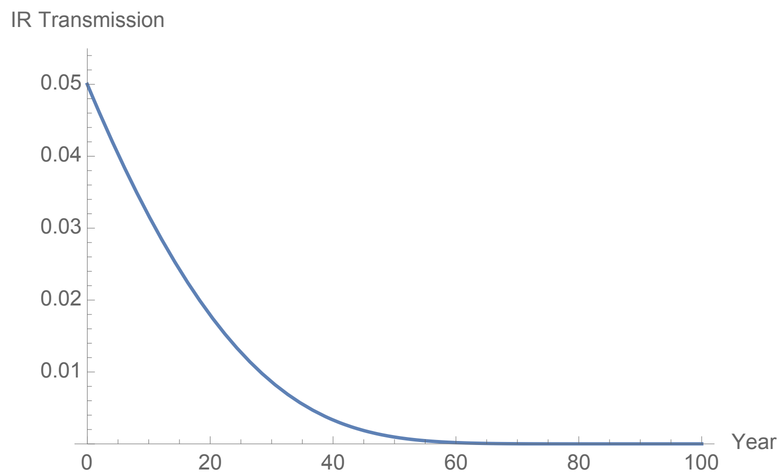

Figure 10: Evolution of the IR transmission coefficient with increasing injection of CO_2.

Temperature expressions

$$
\begin{aligned}
T_s &= 277\left(\frac{1}{0.67+0.05e^{-\frac{45}{44}e^{-\frac{t}{100}}\left(e^{0.04t}-1\right)}}\right)^{0.25}, \\
T_a &= 264\left(\frac{1.4-0.05e^{-\frac{45}{44}e^{-\frac{t}{100}}\left(e^{0.04t}-1\right)}}{0.67+0.05e^{-\frac{45}{44}e^{-\frac{t}{100}}\left(e^{0.04t}-1\right)}}\right)^{0.25}.
\end{aligned}
$$

The evolution of temperatures are shown in Fig. 11.

For a short time compare to the t_{net} ($t \ll t_{\text{net}}$), the simple model we employed gives roughly the same trend of atmospheric CO_2 concentration as IPCC result. The rise in temperature estimated in our model, due to the increase in carbon content in atmosphere, is larger than published results.

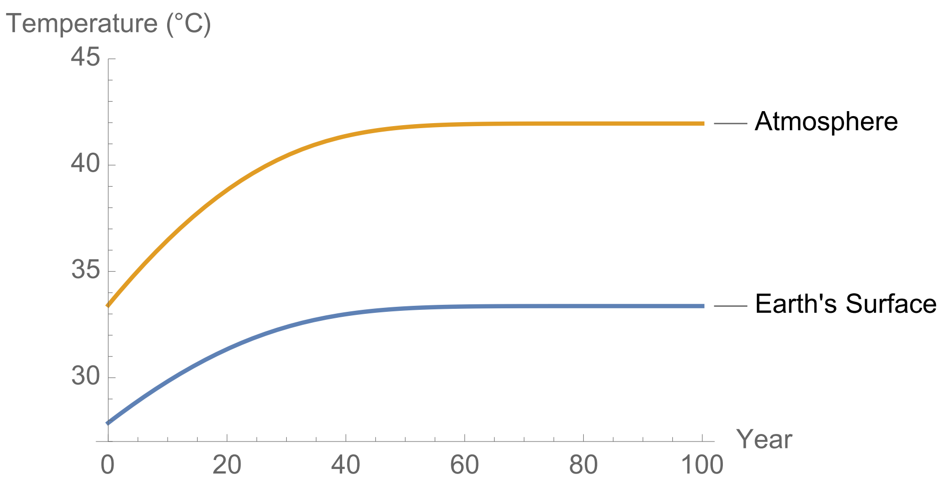

Figure 11: The evolution of temperatures with increasing CO_2 injections