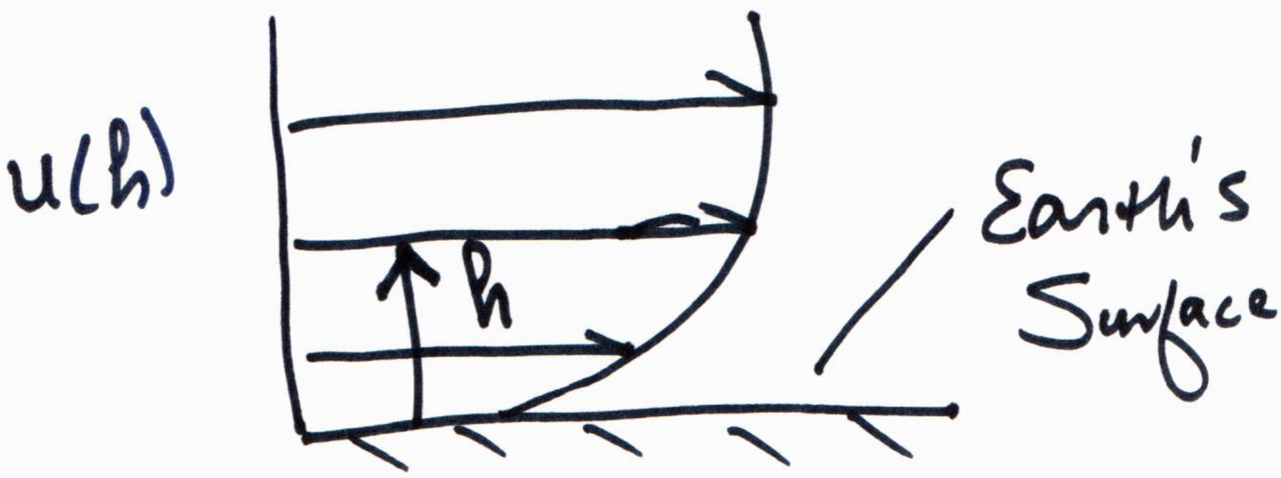
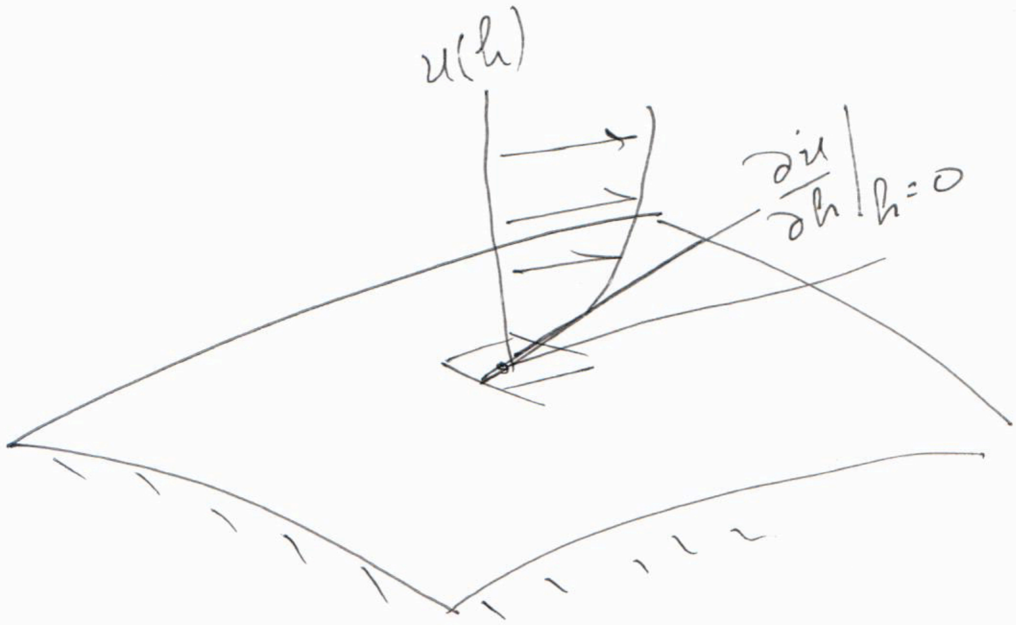
Average Wind Speed Profile in Atmospheric Boundary Layer



$$\frac{u(h)}{w} = 2.5 \ln\left(\frac{h}{r}\right) + 5.5$$

w ~ "skin friction velocity"

r ~ surface roughness



shear stress at $h=0$:

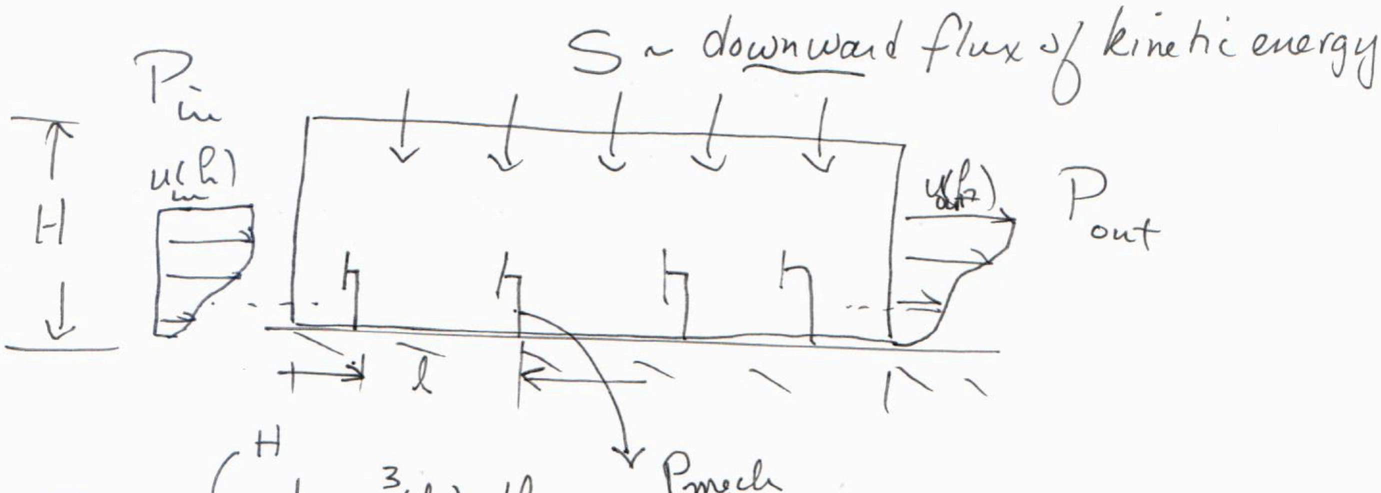
$$\tau(h)\Big|_{h=0} = \mu \frac{\partial^2 u}{\partial h^2}\Big|_{h=0}$$

commonly parameterized in terms of w:

$$\tau\Big|_{h=0} \equiv \rho w^2$$ definition of $\underline{w}$

~~In general, the wind speed can vary from the mean value, u. Let instantaneous speed be given as v. Then the probability that v > u~~

Side-on View of Control Volume:



$$P_{in} = \int_0^H \frac{1}{2}\rho\, u_{in}^3(h)\, dh$$

$$P_{out} = \int_0^H \frac{1}{2}\rho\, u_{out}^3(h)\, dh$$

Let's assume fully developed boundary layer response $\rightarrow u_{in}(h) \simeq u_{out}(h)$

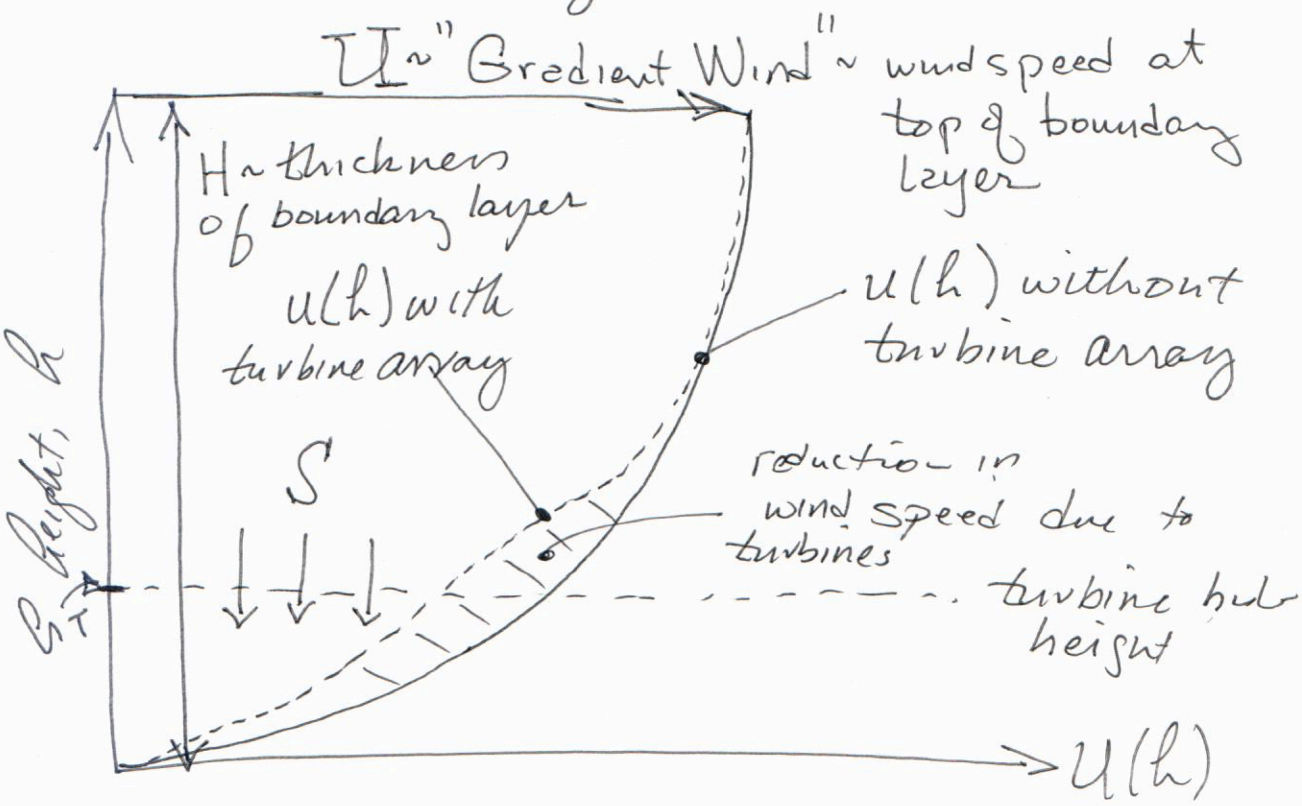
1st law gives (per unit depth)

$$S \cdot l + P_{in} = P_{out} + P_{mech}$$

but since $u_{in}(h) = u_{out}(h) \rightarrow P_{in} = P_{out}$; $\boxed{S \cdot l = P_{mech}}$

Key Points in Following Analysis:

- Assuming boundary is turbulent & we have a well-developed perturbation due to turbine array:



shear stress at surface, $h=0$, is $\tau(h)$

$$\tau(h)\Big|_{h=0} = -\mu \frac{\partial^2 u(h)}{\partial h^2}\Big|_{h=0}$$

S ~ denotes downward flux of wind K.E.

for a turbulent boundary layer, downward flux of kinetic energy:

$$S \simeq \frac{6}{\pi} \rho w^2 u \sim 2 \rho w^2 u$$

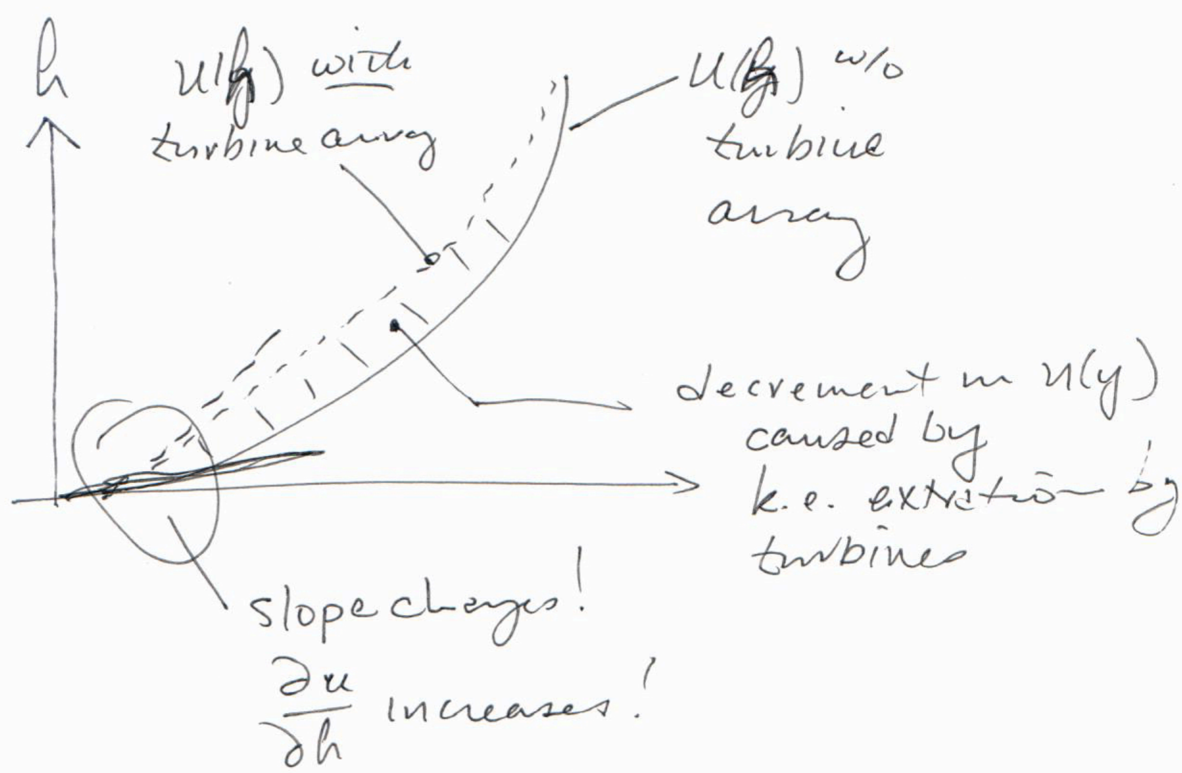
"skin friction velocity"

$u \sim$

Suppose we now install an array of N turbines over a land area, A. Each turbine sweeps out a frontal turbine area A_t. We can then define the ratio of areas, λ, as

$$\lambda \equiv \frac{N A_t}{A}$$

Now we will argue that the introduction of the turbine array perturbs the boundary layer flow pattern as



change in $\frac{\partial u}{\partial h}$ $\Longrightarrow$ increase in sheer stress, τ, at $h=0$

Some take $\delta\tau \propto \lambda$

to make units work:

$$\delta\tau = \lambda \cdot \frac{1}{2}\rho u^2$$

but we previously had defn of ω as

We also have def'n of skin friction velocity, ω:

$$\tau \equiv \rho \omega^2.$$

Take $\delta(\,)$:

$$\delta\tau = 2\rho\omega\,\delta\omega$$

Equate two expressions for $\delta\tau$:

$$2\rho\omega\,\delta\omega = \lambda\,\tfrac{1}{2}\rho u^2$$

known from b. l. model

re-arrange:

$$\boxed{\frac{\delta\omega}{\omega} = \frac{\lambda}{4}\left(\frac{u^2}{\omega^2}\right)} \quad \text{or} \quad \lambda = 4\,\frac{\delta\omega}{w u^2}$$

Remember: we want to know how many turbines we can pack into A and still maintain reasonable power production. In other words we want to know by how much $u(h)$ will decrease due to a given amount λ; i.e. $\frac{\delta u(h)}{u(h)} = f(\lambda)$?

~~Thus we are interested in $\lambda P / S$; which is the ratio of power flux thru the turbine array to the rate of downward k.e. flux~~

Use earlier results to write

(average power/unit area of a Rayleigh Wind Speed Distribution)

$$\frac{\lambda P}{S} = \frac{\lambda P_{ave}}{S} = \frac{\lambda \frac{3}{\pi}\rho U^3}{\frac{6}{\pi}\rho \omega^2 U} = \frac{\lambda}{2}\frac{U^2}{\omega^2}$$

from earlier result we had

$$\frac{\delta\omega}{\omega} = \frac{\lambda}{4}\frac{U^2}{\omega^2}$$

and so we can write

$$\frac{\lambda}{2}\frac{U^2}{\omega^2} = 2\,\frac{\delta\omega}{\omega}$$

In order that the turbines not slow the wind by a large amount we assume

$$\frac{\lambda P_{ave}}{S} \ll 1$$

this then implies that

$$\frac{\lambda}{2}\frac{u^2}{w^2} \ll 1 \quad \text{and} \quad \frac{\delta w}{w} \ll 1.$$

to proceed further now we need to consider the velocity profile in the boundary layer.

It is common to define the wind speed at the "top" of the boundary layer, where $h = H$, as U: $U \sim$ "Gradient Wind"

$$\frac{U}{w} = 2.5 \ln \frac{H}{r} + 5.5$$

[n.b. we are evaluating $\frac{u(h)}{w}$ at height, H, so high that $U = u(H) = \text{const.}$]

It is reasonable to assume that the height, H, of boundary layer is proportional to shear stress at $h \simeq 0$ & thus to w; i.e. $H \propto w$.

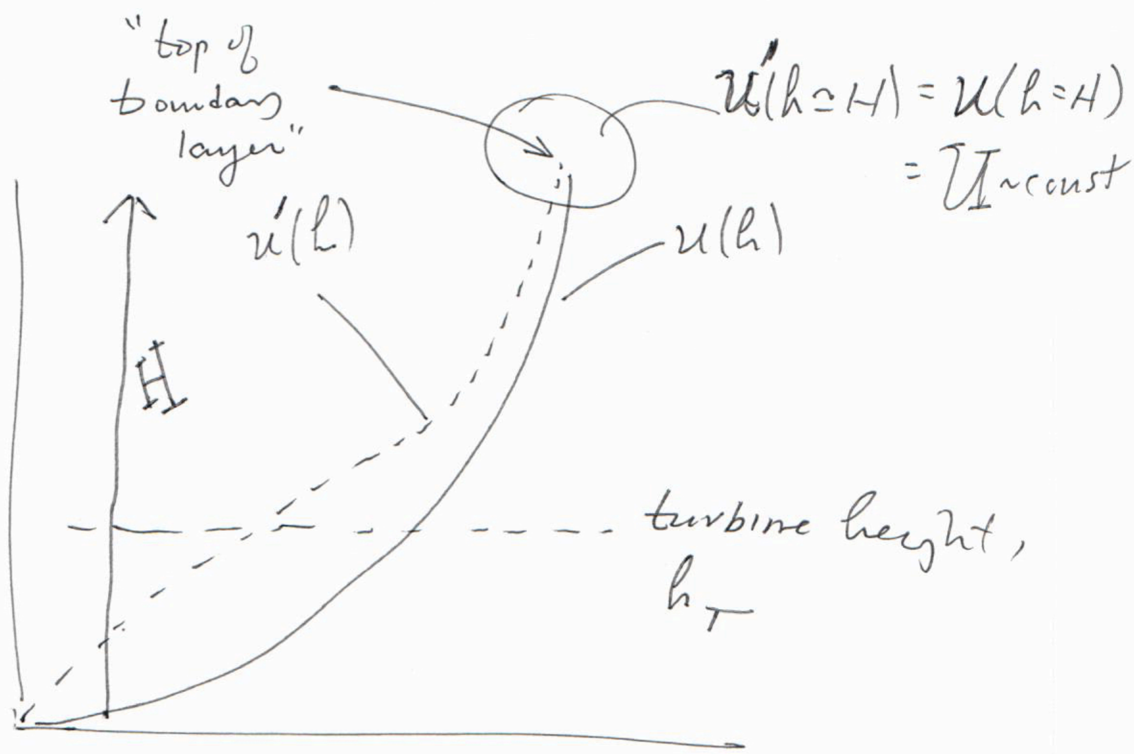
thus $$\frac{\delta H}{H} = \frac{\delta w}{w}.$$

Now... when we introduce the turbine array we get changes in:

δw ~ increase in skin friction velocity since $\delta\tau\big|_{h=0} \neq 0$

δr ~ change in effective surface roughness

but... $\delta U = 0$ since by defin U is measured at "top" of boundary layer, away from region where turbines cause a perturbation to $u(h)$:



using $\frac{U}{w}$ expression we take $\delta(\,)$ to both sides:

from b.l. profile & defn of $U(H)$:

$$\delta\left(\frac{U}{\omega}\right) = \delta\left[2.5 \ln \frac{H}{r} + 5.5\right]$$

or

$$-\frac{U}{\omega^2}\,\delta\omega = 2.5\,\cancel{\frac{\delta H}{H}}\;(\delta\omega/\omega) - \delta \ln r$$

[n.b. U = const but $H \to H + \delta H$ w/ turbine array]

re-arrange:

$$\delta \ln r = \left(1 + 0.4\,\frac{U}{\omega}\right)\frac{\delta\omega}{\omega}$$

Now in the lower region of the boundary layer we have

$$\delta u(h) = \delta\left(\omega\left(2.5 \ln \frac{h}{r} + 5.5\right)\right)$$

Carry out rhs operation:

$$\delta u(h) = \delta\omega\left(2.5 \ln \frac{h}{r} + 5.5\right) + \omega(2.5)\,\delta\left(\ln \frac{h}{r}\right)$$

$$= \delta\omega\,\frac{u(h)}{\omega} - 2.5\,\omega\,\delta(\ln r)$$

Using earlier expression for $\delta(\ln r)$ gives

$$\delta u(h) = \delta\omega\,\frac{u(h)}{\omega} - 2.5\,\omega\left(1 + 0.4\,\frac{U}{\omega}\right)\frac{\delta\omega}{\omega}$$

which can be re-arranged to give

$$\delta u(h) = \delta w \left[\frac{u(h) - U}{w} - 2.5 \right]$$

~~E ~ 2.5 [1~~

now from boundary $u(h)$ eq'n & def'n of U we have

$$\frac{u(h) - U}{w} = 2.5 \ln\left(\frac{h}{r}\right) + 5.5$$

$$- \left[2.5 \ln\left(\frac{H}{r}\right) + 5.5 \right]$$

$$= 2.5 \left[\ln\left(\frac{h}{r}\right) - \ln\left(\frac{H}{r}\right) \right]$$

$$= 2.5 \ln\left(\frac{h}{H}\right)$$

and thus

$$\delta u(h) = \delta w \left[\ln\left(\frac{h}{H}\right) - 1 \right] (2.5)$$

$$= -2.5\, \delta w \left[1 + \ln \frac{H}{h} \right]$$

re-arrange:

$$\frac{\delta u(h)}{u(h)} = -2.5\left(1 + \ln\frac{H}{h}\right)\frac{\delta\omega}{\omega}\,\frac{\omega}{u(h)}$$

but earlier we had

$$\frac{\delta\omega}{\omega} = \frac{u(h)^2}{\omega^2}\,\frac{\lambda}{4}$$

$\therefore$

$$\frac{\delta u(h)}{u(h)} = -2.5\left(1 + \ln\frac{H}{h}\right)\frac{u^2(h)}{\omega^2}\,\frac{\lambda}{4}\,\frac{\omega}{u(h)}$$

or

$$\boxed{\frac{\delta u(h)}{u(h)} = -2.5\left(1 + \ln\frac{H}{h}\right)\frac{u(h)}{\omega}\,\frac{\lambda}{4}}$$

this tells us change in b.l. velocity profile, $\delta u(h)/u(h)$, in terms of unperturbed profile [known] and turbine area ratio, λ!

We can now estimate change in turbine power and in downward flux of wind kinetic energy:

the average turbine power is

$$P_{ave} = \frac{3}{\pi} \rho\, U^2\Big|_{h=h_T} \;; \quad \text{here } U \text{ is wind speed at hub height, } h = h_T$$

$$\therefore \quad \delta P_{ave} = \frac{3}{\pi} \rho\, 2 U \delta U$$

& we can write

known from above result!

$$\boxed{\frac{\delta P_{ave}}{P_{ave}} = 3\left(\frac{\delta U}{U}\right)}$$

we had: S given earlier. Thus

$$S \simeq \frac{6}{\pi} \rho\, w^2 u$$

change in $u(h)$ in b. l.

$$\therefore \quad \boxed{\frac{\delta S}{S} = 2\frac{\delta w}{w} + \frac{\delta u}{u}}$$

change in stress at earth's surface

found earlier: $\boxed{\frac{\delta w}{w} = \frac{\lambda}{4}\frac{u^2(h)}{w^2}}$

We can now estimate the effect of a large turbine array on wind speed & thus effective power production

<u>Specify</u>:

- turbine height, h_t
- the value of wind speed at top of the boundary layer, $\underline{U}$ ~ "Gradient Wind"

<u>~~Compute~~</u>:

- Surface Roughness Parameter, r

  n.b. $r \simeq 0.6$ m ~ on smooth surface (Ocean)

  $r \simeq 10$ m ~ rough topography (e.g. land)

- Skin friction velocity, ω

  $\omega \sim 0.5$ m/sec ocean

  $\omega \sim 0.6$ m/sec land

Compute:

$\frac{u(h_T)}{\omega}$ ~ from b.l. profile model

~~Max Allow~~

$\left(\frac{\delta\omega}{\lambda\omega}\right)$, $\frac{\delta u}{u\lambda}$, $\frac{\delta S}{\lambda S}$ from expression above

for max allowable $\frac{\delta P}{P}$ ⇒ find max $\frac{\delta u}{u}$

$$\lambda_{MAX} = \frac{\max\left(\frac{\delta P}{P}\right)}{\max\left(\frac{\delta u}{\lambda u}\right)}$$

Max. Allowable Power Per Unit Land Area:

$$P_{Land} = \lambda P_{ave}$$

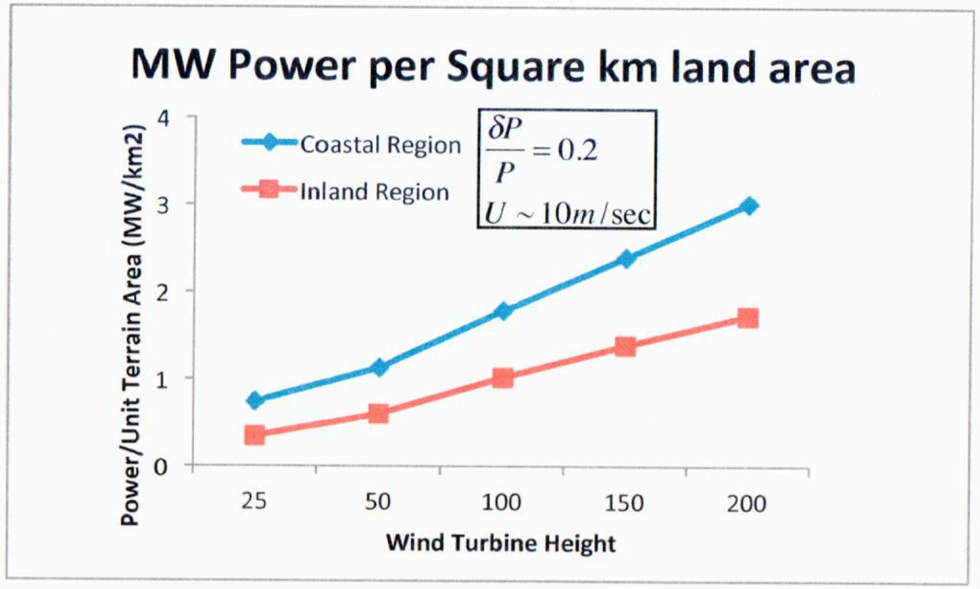
These calcuations based upon analysis contained in the paper
"Limits to Wind Power", R.W. B. Best Energy Conversion v.19 pp.71-72 (1977)

Turbine Axis Height	100 m
Gradient Wind, U	10 m/sec

		Coastal region analysis	Inland region analysis
INPUT QNTY'S			
roughness, r		0.6 m	10 m
Boundary layer thickness, H		400 m	1000 m
skin friction velocity, w		0.5 m/sec	0.64 m/sec
CALCULATED DIMENSIONLESS QNTY'S			
unpert boundary layer	**u/w**	18.2899895	11.2564627
change in skin friction velocity	**dw/(lambda*w)**	83.6309292	31.6769883
change in wind speed at hub height	**du/(lambda/u)**	-27.2783118	-23.2346413
change in downward KE flux	**dS/lambda*S)**	139.983547	40.1193354
Unperturbed windspeed	**u**	9.14499476	7.20413615
	U	10.8778627	10.8882723

for 20% loss in wind power density, dP/P=0.2 we then find lambda:

	Coastal	Inland
dP/P	-0.2	-0.2
allowable du/u	-0.06666667	-0.06666667
Lambda	0.00244394	0.00286928
Average wind power density at turbine hub (W/m^2)	730.334414	357.0402
maximum power/unit terrain area, lambda*P, W/m^2	1.78489641	1.02444792
Maximum power for 1 km^2	1.8E+06	1.0E+06



MW Power/km^2 land area

h (meters)	Coastal Regior	Inland Region
25	0.74167959	0.34564232
50	1.13346426	0.60610165
100	1.78489641	1.02444792
150	2.39520147	1.38762321
200	3.01486256	1.72650009

Let us apply to our earlier estimates:

Land area within 10km of transmission line w $P_{turb} \geq 300 \cancel{MW} W/m^2$ $\quad A_{tot} \simeq 6\times10^5 \, km^2$

if we are willing to suffer ~ 7% reduction in wind speed $\rightarrow \frac{\delta P}{P} \sim 0.20$

then $P' = P - \delta P = 0.8P = 0.8(300) = 240 \, W/m^2$

$\lambda = 2.8\times10^{-3} \longrightarrow$ max power/unit land $= \lambda P'$

$\sim 2.4\,10^2 \cdot 2.8\,10^{-3}$

$\simeq 7\times10^{-1} = 0.7 \, W/m^2$

or $700 \, kW/km^2$

if we use 10% of availble land then

$P^{MAX}_{wind} \simeq 0.1 \lambda P' A_{tot} \simeq 7\times10^{5} \frac{W}{km^2} \cdot 6\times10^4$

$\simeq \underline{42 \, GW}!$