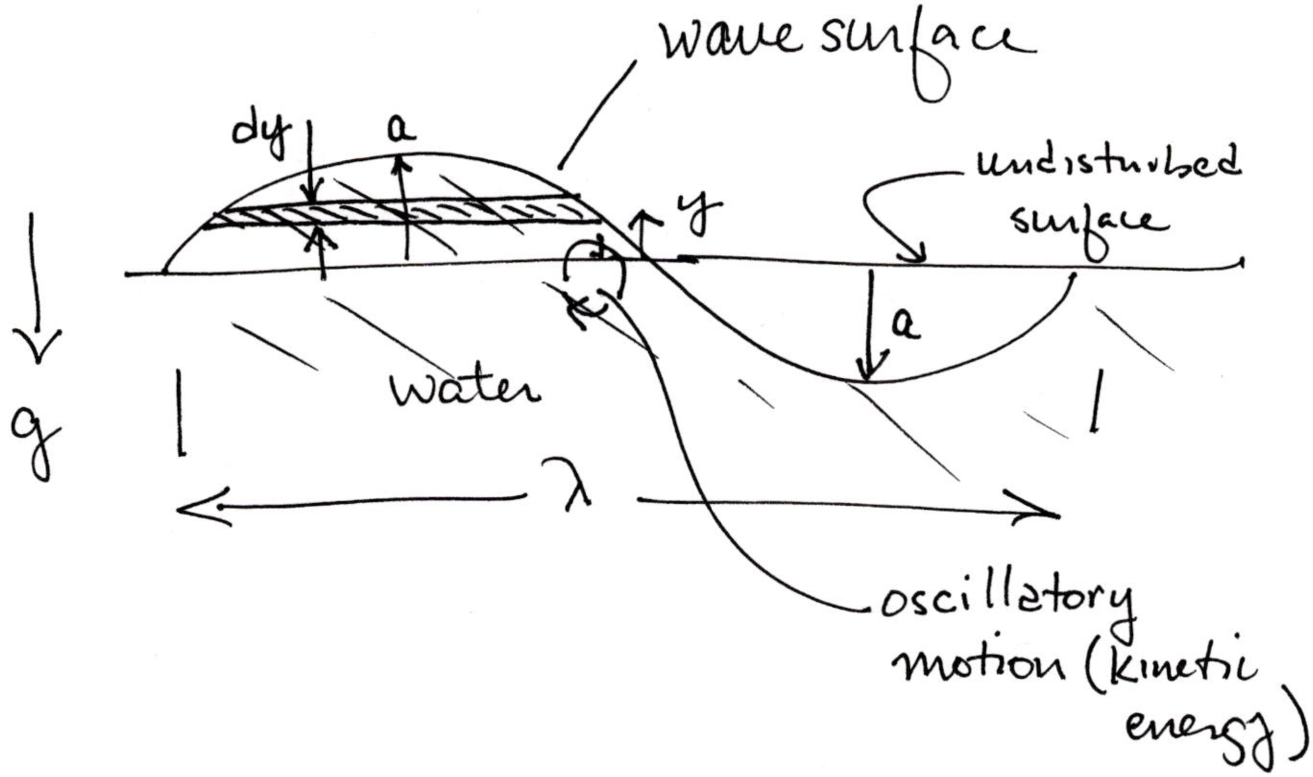
Wave Power - Order-of-Magnitude Estimates



take wave $y(x,t)$:

$$y(x,t) = a \sin(\omega t - kx) \; ; \quad k \sim \text{"wavenumber"} \quad k \equiv \frac{2\pi}{\lambda}$$

phase velocity of wave : $v_{ph} = \dfrac{\omega}{k}$

where $\omega \sim$ frequency (rad/sec)

$k \sim$ wave number (rad/m)

can show from fluid equations

$$v_{ph} = g/\omega \qquad g \sim \text{Earth's gravitational acceleration}$$

note: $v_{ph} = v_{ph}(\omega)$

Group Velocity, $v_{gr} = \partial\omega/\partial k$

Now: Using v_{ph} expressions we can write

$$\frac{g}{\omega} = \frac{\omega}{k} \Longrightarrow gk = \omega^2 \quad \text{"dispersion relation"}$$

We wish to find the amount of total energy in the wave, $U_{tot} = U_{pot} + U_{K.E.}$

It turns out that can show

$$U_{k.e.} = U_{pot}.$$

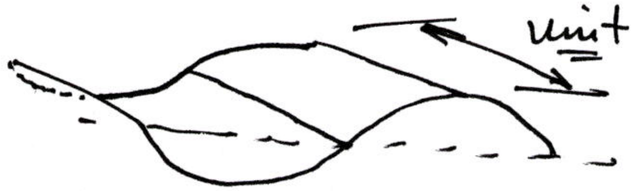
Thus we just need U_{pot}.

Consider incremental potential energy of the element dy:

$$dU_{pot}(y) = \rho g y x \, dy$$

where x denotes horizontal length of wave crest at position y.

[note: this is energy per-unit depth into the paper:



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w/o loss of generality can consider $t=0$.
then we have

$$y(x) = a \sin(kx) = a \sin \frac{2\pi}{\lambda} x$$

we solve for $x = x(y)$:

$$x = \frac{\lambda}{2\pi} \sin^{-1}(y/a) \, dy$$

Use this in dU_{pot}:

$$dU_{pot}(y) = \rho g y \frac{\lambda}{2\pi} \sin^{-1}(y/a)\, dy$$

and the total ~~kinetic~~ potential energy of the wave crest is then

$$U_{pot}\Big|_{crest} = \int_0^a dU_{pot} = \rho g \frac{\lambda}{2\pi} \int_0^a y \sin^{-1}(y/a)\, dy$$

carry out integral to find

$$U_{pot}\Big|_{crest} = \rho g \frac{\lambda a^2}{16} \quad \text{for region } 0 < x < \frac{\lambda}{2}$$

the trough region has an identical potential energy. ~~Thus~~ [Why?] Thus U_{pot} is given as

$$U_{pot} = \rho g \frac{\lambda a^2}{8}$$

Now we have $U_{k.e.} = U_{pot}$ and thus U_{tot} is

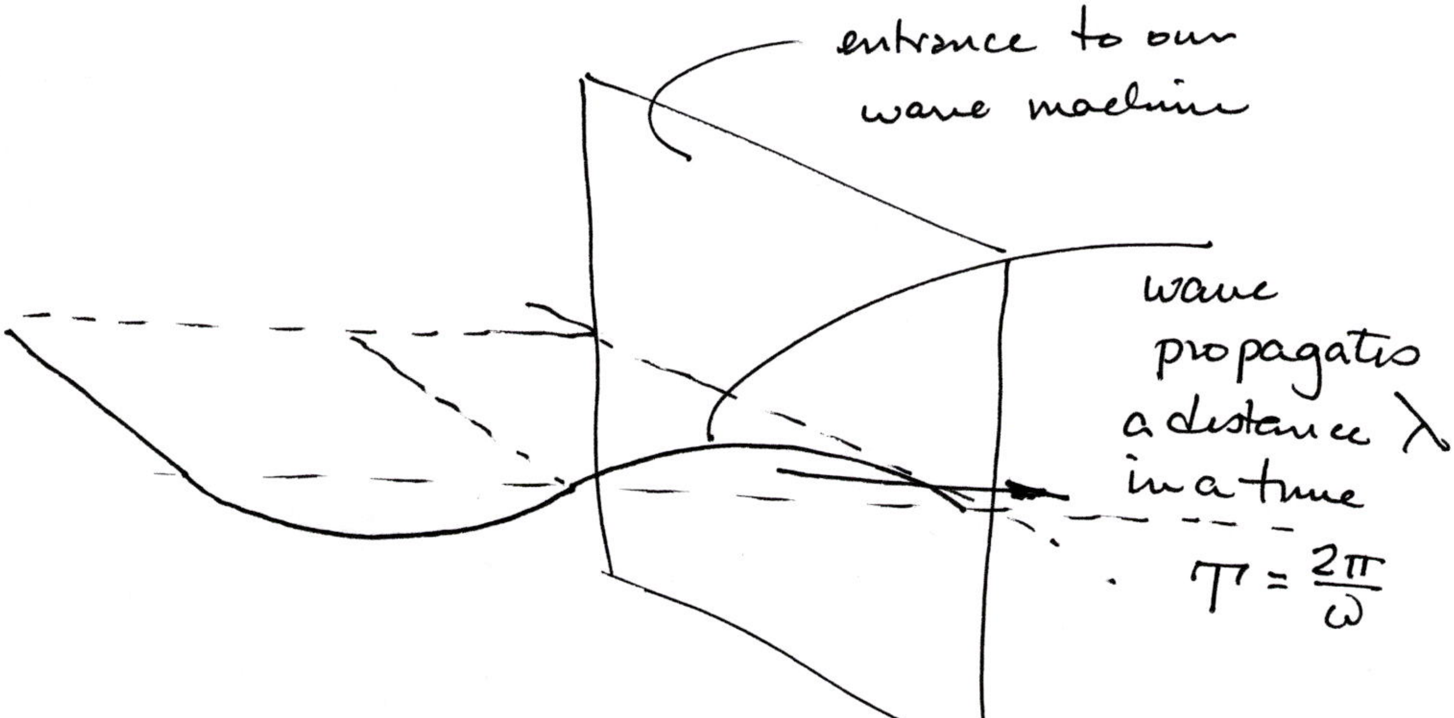
$$U_{tot} = \rho g \frac{\lambda a^2}{4}$$

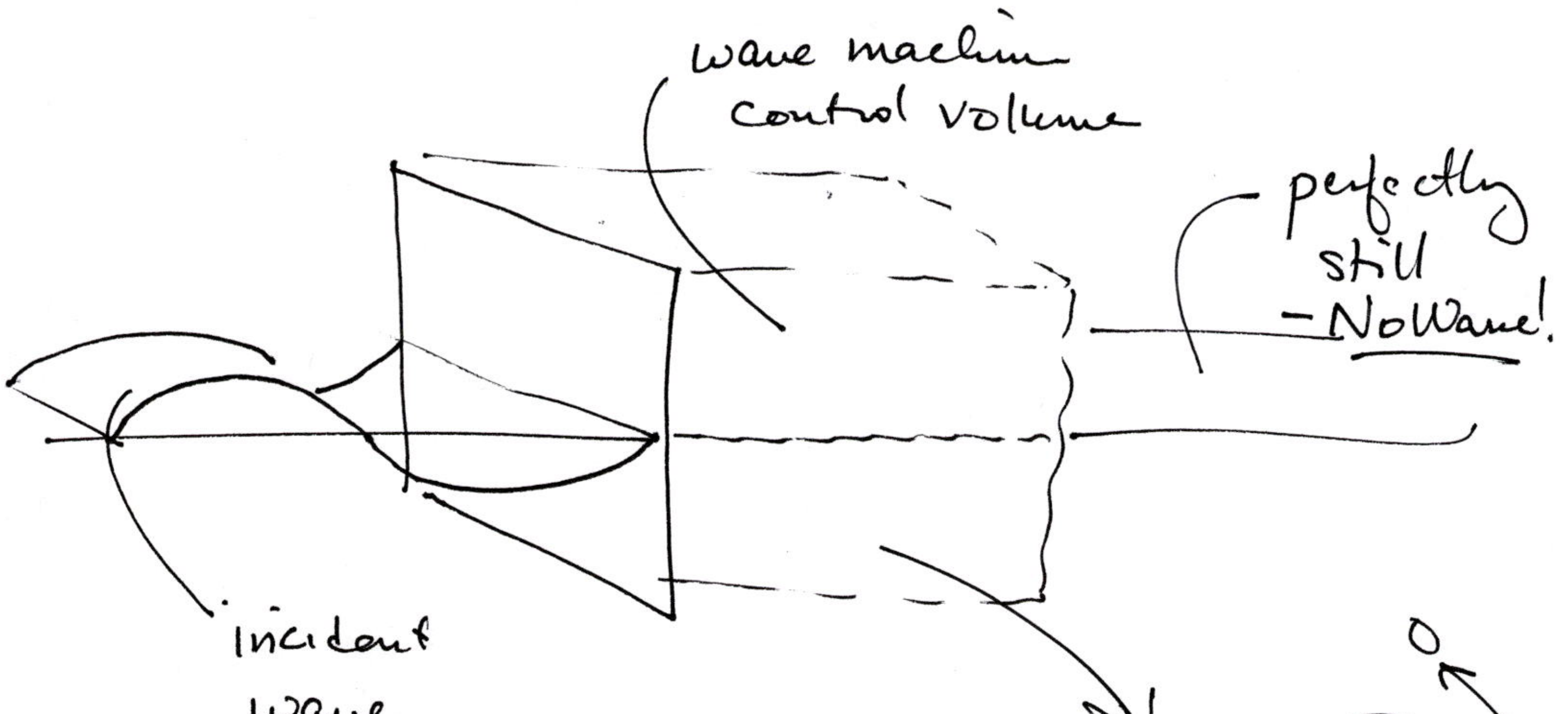
[n.b. units:

$[U_{tot}] \sim J/m \sim \frac{N\text{-}m}{m} \sim N$

$\left[\rho g \frac{\lambda a^2}{4}\right] \sim \frac{kg}{\cancel{m^3}} \frac{m}{s^2} \cancel{m^3} \sim N$ ✓]

Now we wish to estimate maximum available <u>Power</u> in the wave.





$$W_{out} = P_{in} - \overset{0}{\cancel{P_{out}}}$$

$$P_{in}\left(\begin{matrix}\text{power}\\ \text{per}\\ \text{unit length}\end{matrix}\right) = \frac{U_{tot}}{T}$$

$$T = \frac{2\pi}{\omega}\ ; \qquad \omega^2 = gk = g\left(\frac{\lambda}{2\pi}\right)^{-1} \Longrightarrow \cancel{\omega = \sqrt{\frac{g\lambda}{2\pi}}}$$

$$\omega = \sqrt{\frac{2\pi g}{\lambda}}$$

$$\therefore \quad \boxed{P_{in} = \rho\, g^{3/2}\, \frac{\lambda^{1/2} a^2}{4\sqrt{2\pi}}}$$

Estimate Values:

$$T = 10 \text{ sec} \qquad a = 1 \text{ m}$$
$$\rho = 10^3 \text{ kg/m}^3 \qquad g \sim 10 \text{ m/s}^2 \quad \Rightarrow \mathbb{P} \simeq 40 \frac{\text{kW}}{\text{m}}$$

or $\mathbb{P} \simeq 40 \text{ MW/km}$

Consider applying to a land mass of area A surrounded by ocean:
let coastal circumference length be C

Suppose we install wave power devices around the area. ~~Q:~~ Let power demand be $P_d \propto \text{Area} \sim L^2$

Q: How does $\dfrac{P_{wave}^{max}}{P_d}$ scale with land ~~or~~ area?

$$\frac{P_{wave}^{max}}{P_d} \sim \frac{\mathbb{P} \cdot C}{A} \sim \frac{\overset{\text{fixed by wave physics}}{\mathbb{P}} \cdot L}{L^2} \propto \frac{1}{L}$$

i.e. if L is small enough could get

$$\frac{P_{wave}^{max}}{P_d} \to 1 \Longrightarrow \text{100\% wave power}$$

but for large L, $\dfrac{P_{wave}^{max}}{P_d} \ll 1$

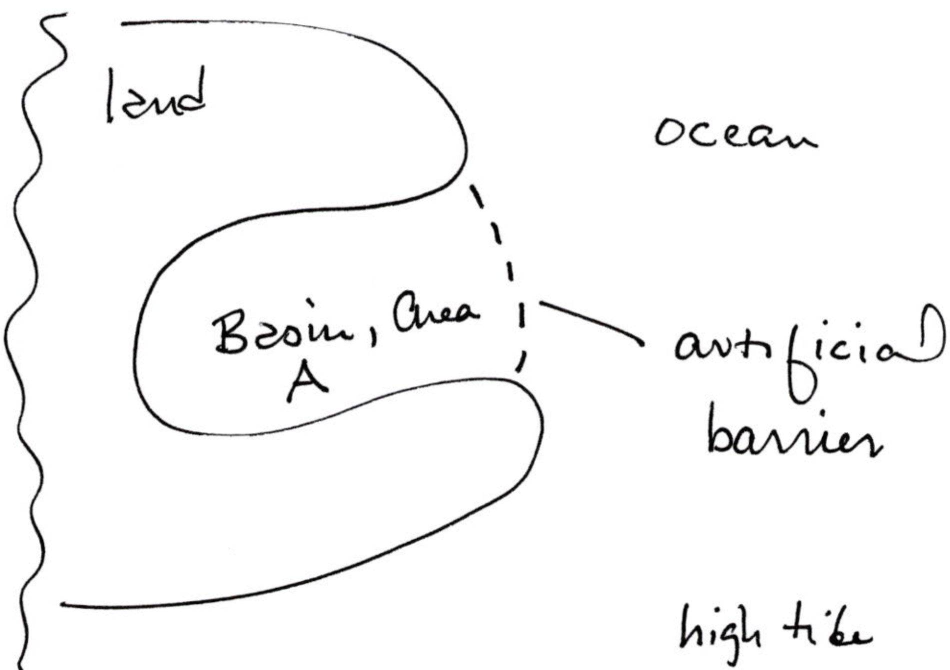
i.e. wave power is small total contributor for large areas (e.g. continents)

could be locally important for small areas (a.k.a. Islands)

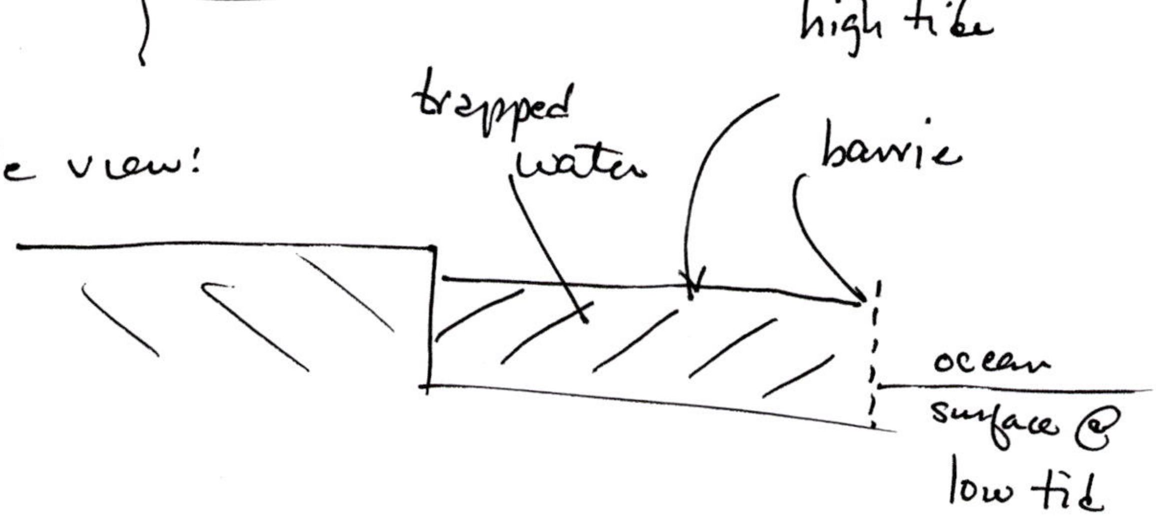
Tidal Power

see schematic on next page

top view:



side view:



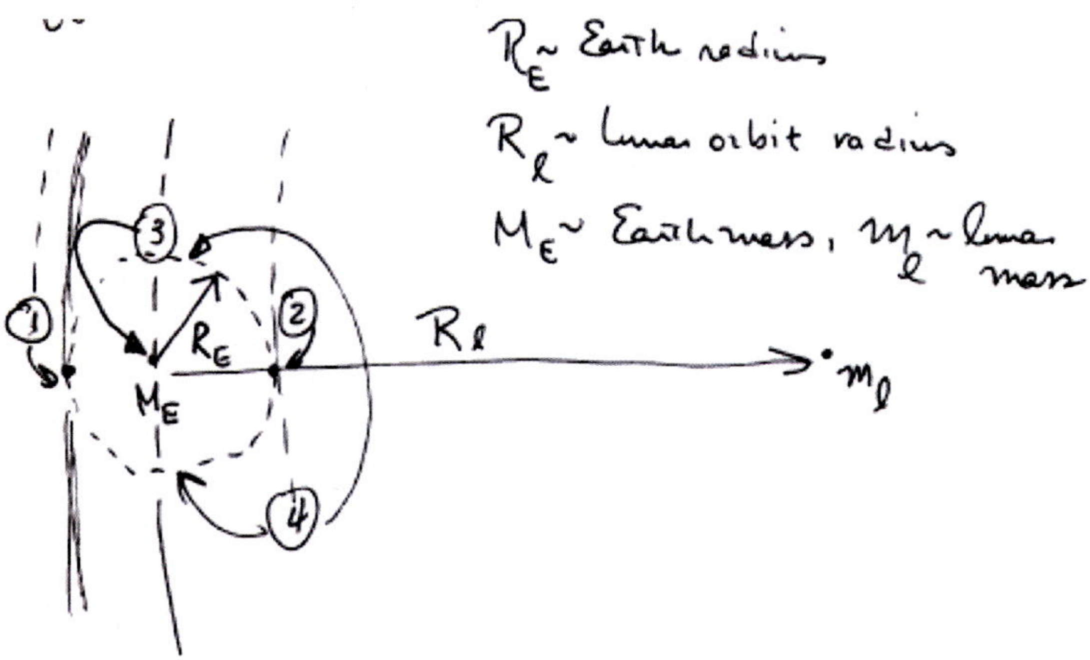


Figure 8.2: Schematic of the Earth-moon system showing the point away from the moon (point 1) which experiences a net gravitational acceleration that is slightly less than that experiences at the point closest to the moon (point 2). Points 3 and 4 experience a net gravity acceleration that is intermediate to that of points 1 and 2.

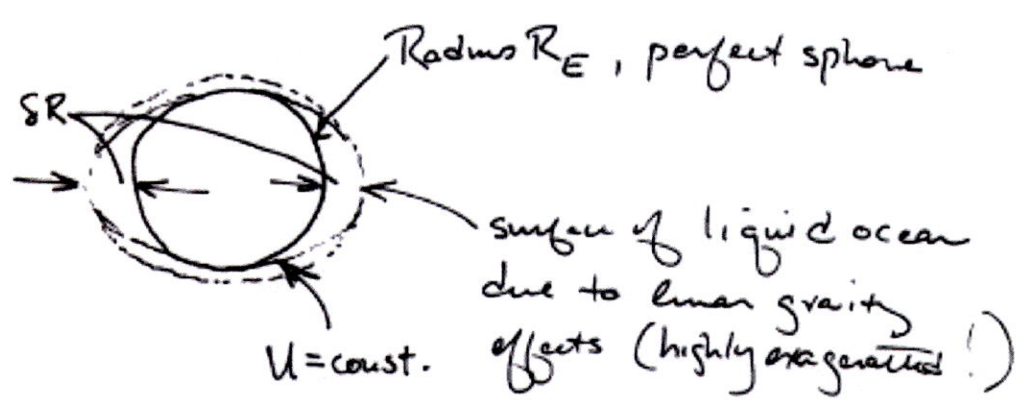


Figure 8.3: The surface of a liquid within a gravitational potential well will lie at an equipotential. For the Earth's oceans, the liquid water surface will acquire a slight deviation away from spherical due to the variation in the net gravitational acceleration arising from the moon (and to a lesser extent, the sun).

Refering to Figure 8.2, the difference in the force on a fluid parcel of mass m due to the lunar mass, m_l, at positions 1 and 2 on the Earth is given as

$$F_2 - F_1 = \frac{mm_lG}{\left(R_l - R_E\right)^2} - \frac{mm_lG}{\left(R_l + R_E\right)^2}.$$

Normalizing this difference to the lunar gravitational force at position 3 we find the relative variation in the lunar gravitational acceleration on the parcel of mass m to be given as

$$\frac{F_2 - F_1}{F_3} = \frac{1}{\left(R_l - R_E\right)^2} - \frac{1}{\left(R_l + R_E\right)^2}.$$

Since we know that $\frac{R_E}{R_l} \sim \frac{1}{60}$ we then can see that the lunar gravity varies by about +/-3% from point 1 to point 2. The ratio of the average lunar gravitational force on mass m to the Earth's gravitational acceleration is given simply as

$\frac{F_E}{F_3} = \frac{M_E}{m_l}\left(\frac{R_l}{R_E}\right)^2 \sim 80 \times 60^2$. Thus we can find the deviation of the net (i.e. Earth + Moon) gravitational acceleration across points 1-2, normalized to Earth's gravitational acceleration, as $\delta F_{net} = \frac{F_2 - F_1}{F_3}\frac{F_3}{F_E} \approx 1 \times 10^{-7}$. Finally, we note that the potential energy, U, of a unit mass parcel of fluid located at the Earth's surface, is given as $U = F \cdot R$ where F is the acceleration on the unit mass and R is a vertical displacement relative to an arbitrary reference plane. Let us take the reference plane to be the surface of spherical ocean that would occur if there were no lunar tides. We can differentiate this expression to write

$$\partial U = \partial F \cdot R + F \cdot \partial R .$$

However, we now note that in the presence of the lunar tidal effect, the surface of the ocean will still lie on an equipotential, and thus we will have $\partial U = 0$. Thus we can find the displacement of the ocean surface due to the tidal effect as

$$\partial R = -\frac{\partial F}{F} \cdot R .$$

Using the result above, we then find that $\partial R = -\frac{\partial F}{F} \cdot R \approx 1 \times 10^{-7} R_E \sim 1m$, which is consistent with typical tidal heights of ~1-2m.

This tidal variation occurs on a timescale corresponding to the time needed for the Earth to rotation approximately ¼ of a turn, i.e. about 6 hours. In some special geographic regions, the ocean water is partially enclosed in a basin. If the period of oscillation, or "sloshing" of the water within this basin is close to this period, then a resonant oscillation can be setup. In this case, much larger tidal oscillations (sometimes up to 10m in a few locations on the Earth) can develop. However, typically the values are of order of a few meters at most.

Tidal power systems are then arranged as follows. A basin region is enclosed from the open sea by the construction of a barrier between the open sea and the basin. This barrier is designed to allow the incoming tidal waters to flow into the basin. Then, when the tidal reverses and the flow begins to move out to sea, the basin traps or retains the water within the basin. As the open sea level then recedes, the resulting height differential can be used to create a potential energy difference. The trapped basin water can then flow through a suitable mechanism that extracts the potential energy and converts it into useful form (usually electricity). Obviously this scheme has a temporal variation that oscillates. If the basin has a surface area, A, and we have a tidal height, h, then the basin volume is obviously V=Ah. This mass has a potential energy, U, given as $U_{pot} = \rho g A h^2$. This potential energy can be extracted on a tidal period T~6 hours. Taking into account the oscillatory nature of the process then gives an average power extraction rate P as

$$\boxed{P_{ave} \approx \frac{1}{2}\frac{\rho g A h^2}{T}.}$$

Let us now make a numerical estimate of the average power that is available from this scheme. Let us assume that the basin has a surface area of 1 km^2. For tidal heights in the range of [1,10] meters, we then estimate $P_{ave} \approx 3\times10^5 - 3\times10^7\ MW/km^2$. The larger values would only hold in few special geographic locations around the world; the intermediate and smaller values could be achieved in most coastal regions. Keeping in mind the anticipated future world energy demand, we estimate that to produce 1 TW of average power we would then require tidal basin areas ranging from $3\times10^6 km^2$ (for height h=1m) to $3\times10^5 km^2$ (for tidal height h=3m). We note that the largest proposed tidal power system (proposed for the Kamchatka peninsula area in the Far East of Russia) has an area of ~20,000 km^2 with a tidal height h=9m, and an estimated average power of ~100 GW (i.e. 1% of estimated future carbon free power demands). The largest existing tidal power system, located in France, has an average power capacity of about 200 MW. Thus it would seem that tidal power, while using well established physical principles and technologies, and which produces power on a highly predictable schedule, nonetheless is limited