

Diffusion-induced wrinkling instability in a circular poroelastic plate

Kai Li,¹ Kewei Ding,¹ and Shengqiang Cai^{2,a)}

¹Department of Civil Engineering, Anhui University of Architecture, Hefei, Anhui 230601, People's Republic of China

²Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, California 92093, USA

(Received 15 March 2013; accepted 6 June 2013; published online 19 June 2013)

A poroelastic material can imbibe solvent and swell. When the material swells inhomogeneously or swells under external constraints, stresses can develop inside the material. The stresses can trigger mechanical instabilities in the material or even break the material, which have been often observed in experiments. In this paper, we study the wrinkling instability of a circular poroelastic plate, in the process of solvent molecules migrating into the plate from the edge. The critical conditions for the initiation and disappearance of wrinkles in the plate are presented. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4811753>]

A poroelastic material can imbibe solvent and swell. Stresses will develop when the material swells under external constraints. For a free-standing homogeneous poroelastic material, in equilibrium state, the swelling ratio is uniform and the stress in the material is zero. In transient state, however, due to the inhomogeneous swelling, the stresses are finite in the material, which can trigger mechanical instabilities in the material or even break the material.¹⁻⁴

A variety of instability patterns in a free-standing poroelastic material, induced by the transient stress field, have been observed in experiments.⁵⁻⁹ The instability patterns usually disappear when the porous material reaches equilibrium state. Fig. 1 shows an example of the swelling process of a free-standing spherical gel. Water molecules can migrate into the gel through the surface and cause swelling. In the transient state, the surface swells more than the interior, resulting in instability patterns. As water migrates in further, the surface pattern coarsens and eventually disappears.

In this paper, by adopting Biot's poroelasticity theory,¹⁰⁻¹² we first calculate how stress fields evolve with time in a circular poroelastic plate, when solvent migrates into it through the edge. By perturbing the transient fields in the plate, we further compute the critical conditions of wrinkling instability in the plate. Ignoring the influence of the wrinkled morphology of the circular plate on the process of the diffusion of water, we can also determine the time of the disappearance of wrinkles in the plate.

A poroelastic plate with radius a and thickness h is sketched in Fig. 2. In the initial state, the poroelastic plate is taken to be homogeneous and stress free with c_0 being the concentration of solvent in the plate (i.e., the number of solvent molecules per unit volume of the material) and μ_0 being the chemical potential of the solvent in the plate. The poroelastic plate is then submerged into a solvent-containing environment with chemical potential μ . If $\mu > \mu_0$, solvent migrates into the material and the material swells. If $\mu < \mu_0$, solvent leaves the material and the material shrinks.

We assume both the top and bottom surfaces of the plate are impermeable to the solvent. In experiments, this assumption

is reasonable when thin impermeable layers are coated on the top and bottom surfaces of the plate. The solvent can only migrate into the plate through its edge. Before instability happens, the deformation of the poroelastic plate is axisymmetric and time dependent. Therefore, we can write the displacement in radial direction as $u_r = u_r(r, t)$, the displacement in vertical direction as $u_z = u_z(r, t)$ and the displacement in hoop direction is zero, namely, $u_\theta = 0$. By adopting small deformation assumption, we can obtain the strain field in the plate

$$\varepsilon_r(r, t) = \frac{\partial u_r(r, t)}{\partial r}, \quad (1a)$$

$$\varepsilon_\theta(r, t) = \frac{u_r(r, t)}{r}, \quad (1b)$$

$$\varepsilon_z(r, t) = \frac{\partial u_z(r, t)}{\partial r}. \quad (1c)$$

The concentration of the solvent in the poroelastic plate is also a time dependent field, $c(r, t)$. The number of solvent molecules is conserved, so that

$$\frac{\partial c}{\partial t} + \frac{\partial(rJ)}{r\partial r} = 0, \quad (2)$$

where J is the flux of the solvent in the radial direction.

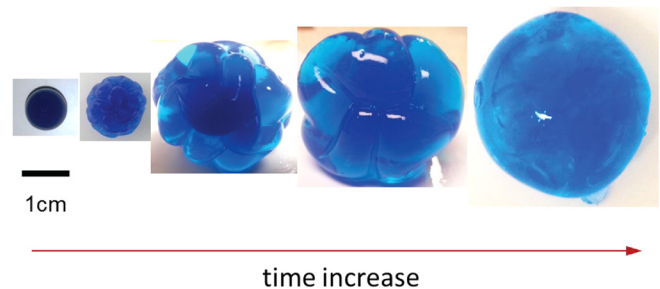


FIG. 1. Swelling process of a spherical gel. Water migrates into the gel from the surface and causes the swelling. In the transient state, compressive stresses develop in the outer part of the gel, which induce complex mechanical instability patterns on the surface. The instability patterns disappear, when the spherical gel reaches equilibrium state.

^{a)}shqcai@ucsd.edu

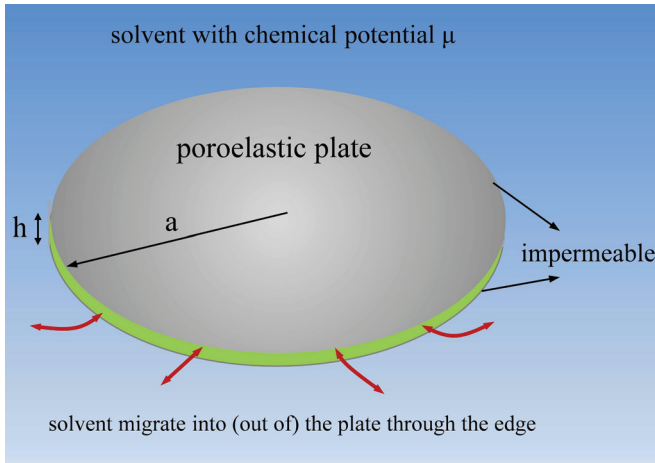


FIG. 2. A poroelastic plate, with radius a and thickness h , contacts with a solvent-containing environment. Initially, the poroelastic plate contains the solvent with the concentration c_0 and chemical potential μ_0 . The chemical potential of the solvent in the environment is μ . The top and bottom surfaces of the plate are both impermeable. The poroelastic plate can only exchange the solvent with the environment through the edge.

The poroelastic plate is in mechanical equilibrium in the process of solvent migration, so that stress field satisfies

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad (3)$$

where σ_r is radial stress and σ_θ is hoop stress.

The poroelastic plate, however, is not in diffusive equilibrium, so that the chemical potential of the solvent in the gel is a time-dependent field, $\mu(r, t)$. The migration of the solvent in the gel is taken to obey Darcy's law

$$J = - \left(\frac{\kappa}{\eta \Omega^2} \right) \frac{\partial \mu}{\partial r}, \quad (4)$$

where κ is the permeability of the poroelastic material, η the viscosity of the solvent, and Ω the volume per solvent molecule.

Without losing generality, we assume that the increase in the volume of the poroelastic material is entirely due to the additional solvent molecules absorbed, namely,

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = \Omega(c - c_0). \quad (5)$$

We can write the equations of state of a poroelastic material by the following:¹¹

$$\sigma_r = 2G \left[\varepsilon_r + \frac{\nu}{1-\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \right] - \frac{\mu - \mu_0}{\Omega}, \quad (6a)$$

$$\sigma_\theta = 2G \left[\varepsilon_\theta + \frac{\nu}{1-\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \right] - \frac{\mu - \mu_0}{\Omega}, \quad (6b)$$

$$\sigma_z = 2G \left[\varepsilon_z + \frac{\nu}{1-\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \right] - \frac{\mu - \mu_0}{\Omega}. \quad (6c)$$

In the above equations, G is the instantaneous shear modulus of the poroelastic material, ν is Poisson's ratio of the poroelastic material in equilibrium state, and the quantity $(\mu - \mu_0)/\Omega$ is known as the pore pressure in the theory of poroelasticity.

Detailed explanation of these quantities can be found in Ref. 11.

As the solvent migrates into or out of the poroelastic plate, the stress in the vertical direction is always zero, i.e., $\sigma_z = 0$. A combination of all the above equations gives the two controlling equations in terms of the radial displacement $u_r(r, t)$ and the solvent concentration $c(r, t)$

$$\frac{\partial c}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c}{\partial r} \right), \quad (7a)$$

$$2 \frac{\partial}{\partial r} \left[\frac{\partial (r u_r)}{r \partial r} \right] = \Omega \frac{\partial c}{\partial r}, \quad (7b)$$

where $D = G\kappa/(1 - 2\nu)\eta$. Eqs. (7a) and (7b) can be solved numerically, subject to the boundary conditions, $\mu(a, t) = \mu$, $\sigma_r(a, t) = 0$ and $u_r(0, t) = 0$, and initial conditions $c(r, 0) = c_0$, $\mu(r, 0) = \mu_0$ and $u_r(r, 0) = 0$. It is noted that both the controlling equations and boundary conditions are linear and the quantity $\mu - \mu_0$ is the only loading parameter in this problem. Consequently, stress, strain, and concentration fields in the plate all linearly depend on the chemical potential $\mu - \mu_0$. To present results, we normalize radial stress, hoop stress, and concentration field in the plate as $\sigma_r \Omega / (\mu - \mu_0)$, $\sigma_\theta \Omega / (\mu - \mu_0)$, and $G \Omega^2 (c - c_0) / (\mu - \mu_0)$. We also normalize the time as Dt/a^2 .

Fig. 3 plots the concentration of the solvent, the radial stress and hoop stress in the plate for several different times. The Poisson's ratio ν is assumed to be 0.3 in the calculation. To interpret the results in Fig. 3, we assume chemical potential of solvent in the environment μ is larger than the initial chemical potential of solvent in the plate μ_0 , so the poroelastic plate swells with solvent migrating into the material. As shown in Fig. 3(a), the concentration of solvent in the plate increases with time till equilibrium. As illustrated in Fig. 3(b), the radial stress in the plate is tensile and reaches maximum in the center of the plate. The magnitude of the radial stress in the center of the plate increases initially and then drops to zero in equilibrium state. The tensile radial stress may cause the fracture of the material. The hoop stress is compressive, close to the edge of the plate, but tensile in the middle of the plate (Fig. 3(c)). The compressive hoop stress may trigger wrinkling instabilities in the plate. In the following, we use linear stability analysis to calculate the critical conditions of mechanical instability in the poroelastic plate.

We denote the out-of-plane deflection in the plate as $w = w(r, \theta)$. The Föppl-von Kármán equations,¹³ governing the deflection of the circular plate, can be written as

$$K \Delta^2 w = \sigma_r h \frac{\partial^2 w}{\partial r^2} + \sigma_\theta h \left(\frac{\partial w}{r \partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right), \quad (8)$$

in which $K = Eh^3/12(1 - \nu^2)$ is the bending stiffness and the differential operator is defined as $\Delta = \partial^2/r^2 + \partial/r\partial r + \partial^2/r^2\partial\theta^2$. The associated boundary conditions are given as $r^2\partial^2 w/\partial r^2 + \nu(r\partial w/\partial r + \partial^2 w/\partial\theta^2) = 0$, $r\partial(\Delta w)/\partial r + (1 - \nu)\partial^2(\partial w/r\partial\theta)/\partial r\partial\theta = 0$ at the free edge $r = a$, and $w = 0$, $\partial w/\partial r = 0$ at the center $r = 0$.

To solve Eq. (8), we set $w = f(r)\cos(m\theta)$ and substitute it into Eq. (8), we obtain that

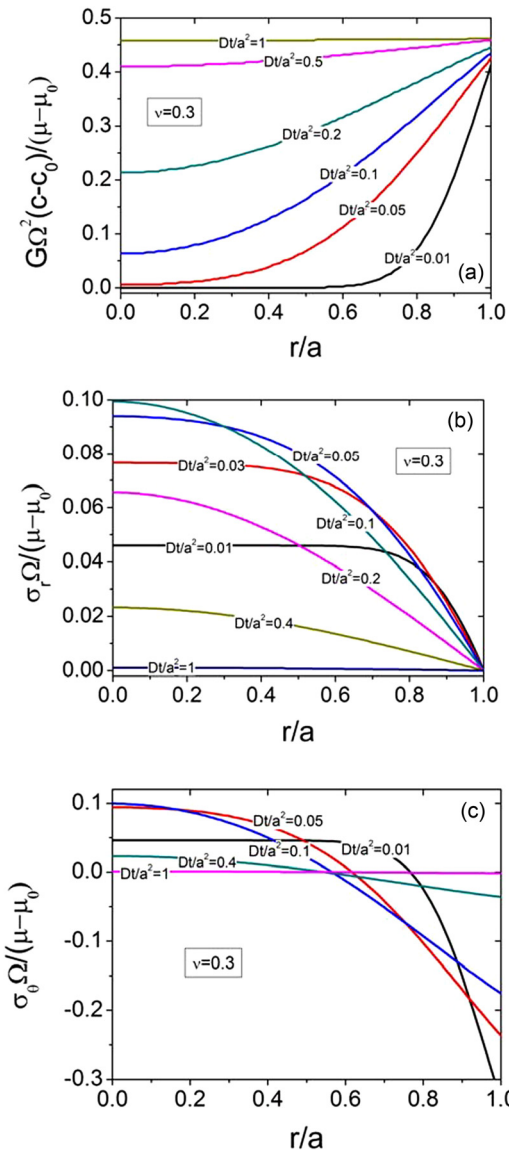


FIG. 3. The distribution of (a) concentration, (b) radial stress and (c) hoop stress in the plate for several times.

$$K\Delta_r^2 f = \sigma_r h \frac{d^2 f}{dr^2} + \sigma_\theta h \left(\frac{df}{r dr} + \frac{m^2 f}{r^2} \right), \quad (9)$$

where $\Delta_r = d^2/dr^2 + d/dr - m^2/r^2$. The corresponding boundary conditions for $f(r)$ become $r^2 d^2 f/dr^2 + \nu r df/dr - \nu m^2 f = 0$, $r^3 d^3 f/dr^3 + r^2 d^2 f/dr^2 - r[1 + (2 - \nu)m^2] df/dr + (3 - \nu)m^2 f = 0$ at the free edge $r = a$, and $f = 0$, $df/dr = 0$ at the center $r = 0$. In Eq. (9), σ_r and σ_θ are the in-plane radial stress and hoop stress, evolving with time as shown in Figs. 3(b) and 3(c). Eq. (9) along with the boundary conditions constitutes an eigenvalue problem. For a fixed chemical potential $\mu - \mu_0$, Eq. (9) can find nontrivial solutions only at specific times, which correspond to the critical conditions of wrinkling instabilities of the poroelastic plate.

Fig. 4 plots the critical time for wrinkling instability of the plate as a function of the chemical potential of the solvent in the environment. The effect of the aspect ratio of the plate a/h and the shear modulus G on the critical conditions of wrinkling instability can be eliminated by normalizing chemical potential as $(\mu - \mu_0)(a/h)^2/G\Omega$. As illustrated in

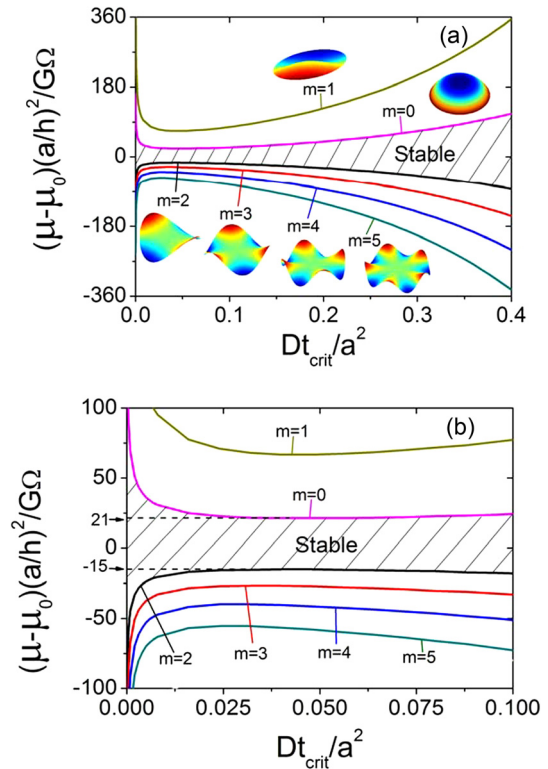


FIG. 4. Critical times for wrinkling instabilities of different modes in the poroelastic plate. For a range of chemical potential, $-15 < (\mu - \mu_0)(a/h)^2/G\Omega < 21$, mechanical instability will not be induced by the transient stress field in the process of solvent migration. Outside the range, two critical times can be found for each mode of wrinkling instability. The first time corresponds to the initiation of wrinkles and the second one corresponds to the disappearance of the wrinkles. (b) enlarges (a) for short time period.

Fig. 4, the critical time depends on the mode of wrinkling instability, m . If solvent migrates into the poroelastic plate and cause the swelling, namely, $\mu > \mu_0$, wrinkling modes with $m \geq 2$ may occur. If solvent migrates out of the plate and cause the shrinking, i.e., $\mu < \mu_0$, wrinkling modes with $m = 0$ or 1 are expected. For a range of the chemical potential (Fig. 4(b)), $-15 < (\mu - \mu_0)(a/h)^2/G\Omega < 21$, wrinkling instability will not be triggered by the transient stress field in the poroelastic plate. Outside the range, at each chemical potential, two critical times can be found for each instability mode. The first critical time corresponds to the initiation of wrinkling instabilities of a particular mode, and the second one corresponds to the vanishing of wrinkles, if the influence of the wrinkled morphology on the field evolution is neglected.

In a summary, the paper studies the conditions of wrinkling instability in a poroelastic plate, induced by the migration of solvent. Depending on the chemical potential difference of the solvent in the environment and in the poroelastic plate at the initial state, wrinkling instabilities of different modes may be triggered by the transient stress field. We also obtain a particular range of chemical potential, in which no mechanical instabilities can be triggered by the transient stress field, in the process of solvent migration.

Kai Li acknowledges the support from the 2010 doctoral startup foundation, and postdoctoral foundation from Anhui University of Architecture. Shengqiang Cai acknowledges the startup funds from the Jacobs School of Engineering at University of California, San Diego.

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