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Gravity induced crease-to-wrinkle transition in soft materials

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Creasing and wrinkling instability are two distinct surface instability modes characterized by localized singular folds and continuous smooth undulations, respectively. In this article, we show that the surface of a soft elastomer may develop wrinkles or creases under compression and the action of gravitational force, depending on the magnitude of gravitational force. Using linear perturbation analysis and numerical calculations, we establish a phase map with respective creasing domain, wrinkling domain and the domain of homogenous deformation. When the gravitational force is small, the surface of the elastomer forms creases when the compressive strain is beyond a critical value, while the surface of the elastomer forms wrinkles under compression when the gravitation force is large. © *2015 AIP Publishing LLC*.
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Wrinkles and creases are two fundamentally distinct mechanical instability modes, which can be often observed in deformed soft materials such as gels and rubbers.[1–3] It has been demonstrated in both experiments and theoretical analyses that when a soft elastic solid is compressed beyond a critical strain, the free surface suddenly forms creases with self-contact.[1,4] In a recent experiment,[5] wrinkles have been observed on the surface of a soft gel under the effect of gravitational force. Those experiments suggest that the competition between elastic energy and gravitational potential energy in a soft solid may determine its surface instability mode-creases or wrinkles.

The effects of gravity on the elastic deformation of a solid can be evaluated by the magnitude of a dimensionless number: $\alpha = \rho g H/\mu$, where ρ is the density, g is the gravity, H is the characteristic size, and μ is the elastic modulus of the solid. When the dimensionless number α is comparable or larger than 1, gravitational force may greatly affect the elastic deformation of solids. An alternative way of evaluating the effects of gravity on the elastic deformation of a solid is through introducing an intrinsic length scale $L_g = \mu/\rho g$. When the characteristic length of the solid is comparable or larger than L_g, the effects of gravity on the elastic deformation of the structure may be significant. For example, when a mountain range is built in the crust, α is large due to the considerable characteristic size (or in another word, the size of the mountain is much larger than its intrinsic length scale L_g). Consequently, gravitational instability can happen in continental lithosphere.[6] Gravitational force also plays important roles in the deformation of soft bio-tissue such as intestinal tissue,[7] which is commonly soft with a Young's modulus ranging from several hundred to several thousand Pa.[8]

In this article, we investigate the conditions for the onset of creases and wrinkles on the surface of the soft elastic solid under compression and subject to the gravitational force. By comparing the onset conditions of creases and wrinkles, we establish a phase map with respective creasing domain, wrinkling domain, and the domain of homogenous deformation.

We first briefly summarize the governing equations of an elastic solid undergoing finite deformation. Deformation gradient of the solid is defined as

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K}, \tag{1}$$

where X_K is the coordinates of a material point of the elastomer in undeformed state and x_i is the coordinate of the same material point in deformed configuration.

Using thermodynamics, the constitutive model of the solid can be specified by a certain free energy density function $W(\mathbf{F})$, namely,

$$S_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}, \tag{2}$$

where S_{iK} is the nominal stress.

With taking account of the gravitational force, the mechanical equilibrium of the solid requires that

$$\frac{\partial S_{iK}}{\partial X_K} + \rho g_i = 0, \tag{3}$$

where g_i is the component of gravity.

Fig. 1(a) sketches the model to be analyzed in the article. A block of an undeformed elastomer with thickness H is taken to be the reference state. The gravity force α and uniaxial prestretch λ^{Pre} are applied to the block of elastomer as shown in Fig. 1(b). The top surface of the elastomer is not allowed to move vertically and the elastomer is assumed to deform in plane strain condition. When the compression or gravitational force is large, homogenously deformed elastomer may bifurcate either into wrinkling state (Fig. 1(c)) or creasing state (Fig. 1(d)) depending on the magnitude of the dimensionless parameter α.

To obtain the critical conditions of wrinkling of the elastomeric block under compression, we adopt linear perturbation analysis.[2] Before perturbation, the deformation in the elastomer is homogenous with prestretch λ^{Pre}, which is given by

$$x_1^0 = \lambda^{\mathrm{Pre}} X_1, \tag{4a}$$

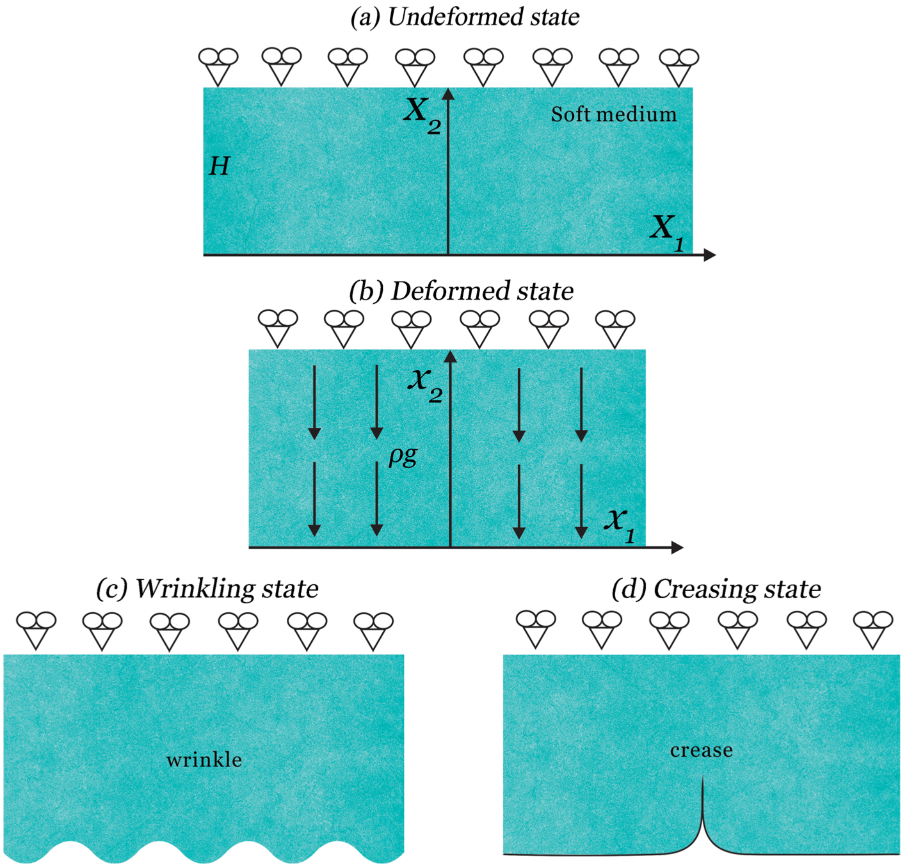

FIG. 1. An elastomer is subject to gravity and uniaxial compression. (a) The undeformed elastomer is taken to be the reference state with thickness H. (b) Under gravity, the elastomer deforms homogenously with pre-stretch λ^{Pre}, but the stress in the elastomer is inhomogeneous. When the compressive strain is large enough, the homogenously deformed state may bifurcate into (c) wrinkling state or (d) creasing state.

$$x_2^0 = X_2/\lambda^{\mathrm{Pre}}. \tag{4b}$$

It is worthwhile to point out the stress in the elastomer is inhomogeneous without perturbation.

Next, we perturb the homogenous deformation by a state of infinitesimal displacement $\tilde{x}_i(\mathbf{X})$ to obtain an inhomogeneous deformation

$$x_i(\mathbf{X}) = x_i^0(\mathbf{X}) + \tilde{x}_i(\mathbf{X}). \tag{5}$$

The corresponding additional deformation gradient $\tilde{F}_{iK}$ and nominal stress $\tilde{S}_{iK}$ caused by the perturbations are

$$\tilde{F}_{iK} = \frac{\partial \tilde{x}_i(\mathbf{X})}{\partial X_K}, \tag{6}$$

$$\tilde{S}_{iK} = \frac{\partial^2 W(\mathbf{F})}{\partial F_{iK}\partial F_{jL}}\tilde{F}_{jL}. \tag{7}$$

The perturbed nominal stress needs to satisfy the force balance equations

$$\frac{\partial(\tilde{S}_{iK})}{\partial X_K} = 0. \tag{8}$$

A combination of Eqs. (6)–(8) gives the governing equations for the infinitesimal displacement $\tilde{x}_i(\mathbf{X})$. The boundary conditions for the perturbations are

$$\tilde{x}_1(X_1, H) = 0, \tag{9a}$$

$$\tilde{x}_2(X_1, H) = 0, \tag{9b}$$

$$\tilde{S}_{12}(X_1, 0) = 0, \tag{9c}$$

$$\tilde{S}_{22}(X_1, 0) = 0. \tag{9d}$$

In this article, we assume the elasticity of the elastomer can be described by Neo-Hookean model, with the free energy density $W(\mathbf{F})$[9,10]

$$W(\mathbf{F}) = \frac{\mu}{2}F_{iK}F_{iK} - \pi(\det(\mathbf{F}) - 1), \tag{10}$$

where μ is the small-deformation shear modulus and $\pi(\mathbf{X})$ is the Lagrange multiplier to enforce the constraint of incompressibility.

To solve the perturbation field, we assume

$$\tilde{x}_1(X_1, X_2) = f_1(X_2)\sin(mX_1), \tag{11a}$$

$$\tilde{x}_2(X_1, X_2) = f_2(X_2)\cos(mX_1). \tag{11b}$$

Substituting Eqs. (11a) and (11b) into Eqs. (6)–(8), we obtains that

$$f_2^{IV} - m^2\left(1 + \frac{1}{(\lambda^{Pre})^4}\right)f_2'' + \frac{m^4}{(\lambda^{Pre})^4}f_2 = 0. \tag{12}$$

The differentiation of f_2 is over X_2 and m is the wavenumber in Eq. (11) and it relates to the wavelength λ of the wrinkle by $\lambda = 2\pi/m$.The ordinary differential Eq. (12), accompanied with the boundary condition, leads to an eigenvalue problem, of which the trivial solution represents the homogeneous state, while the nontrivial solutions correspond to the wrinkling state. The eigenvalue that determines the onset condition of wrinkling can be obtained by solving

$$\det\begin{bmatrix} e^{mH} & -e^{-mH} & \frac{1}{(\lambda^{Pre})^2}e^{\frac{mH}{(\lambda^{Pre})^2}} & -\frac{1}{(\lambda^{Pre})^2}e^{\frac{-mH}{(\lambda^{Pre})^2}} \\ \frac{e^{mH}}{1+(\lambda^{Pre})^4} & \frac{e^{-mH}}{1+(\lambda^{Pre})^4} & \frac{e^{\frac{mH}{(\lambda^{Pre})^2}}}{2} & \frac{e^{\frac{-mH}{(\lambda^{Pre})^2}}}{2} \\ \frac{\alpha\lambda^{Pre}}{mH}+2 & \frac{\alpha\lambda^{Pre}}{mH}-2 & \frac{\alpha\lambda^{Pre}}{mH}+\frac{1+(\lambda^{Pre})^4}{(\lambda^{Pre})^2} & \frac{\alpha\lambda^{Pre}}{mH}-\frac{1+(\lambda^{Pre})^4}{(\lambda^{Pre})^2} \end{bmatrix}=0. \quad (13)$$

The critical strain $\varepsilon_{critical}$, defined as the strain at which nontrivial solutions exist in Eq. (12) for a given gravity α, is plotted in Fig. 2(a). Biot's classical result of the wrinkling in an elastomer under compression are recovered for $\alpha = 0$. With the increase of gravity, less compressive strain is needed to induce wrinkles on the surface of the elastomer. Gravity may even induce wrinkles with certain wavelength on the surface of a pre-stretched elastomer (e.g., when $\alpha > 7$). The reason that gravity can facilitate the formation of wrinkles is the gravitational potential of the solid can be reduced through surface wrinkling.[5]

In the inset of Fig. 2(a), we can also find that when the wavelength of wrinkles is small enough, namely, $\lambda \ll H$ and $\lambda \ll L_g$, the critical strain for wrinkling is independent of the wavelength and the magnitude of α.

As shown in Fig. 2(a), for a certain gravity, critical strain for wrinkles depends on their wavelength. There exists one wavelength of wrinkle requiring smallest compressive strain (or largest tensile strain), which is defined as the critical mode. Wavelength of critical mode is plotted as a function of gravity in Fig. 2(b). The red-cross in Fig. 2(b) is the recent experimental measurement of the wrinkle wavelength on the surface of a soft gel only under the action of gravitational force.[5]

As discussed at the beginning of this article, the surface of a compressed elastomer forms creases instead of wrinkles when the gravitational force is negligible. Our recent researches on creases have shown that the strain for the onset of creases cannot be predicted by linear perturbation analysis.[11,12] Instead, a combination of numerical calculations and energetic analysis, adopted in the previous studies, precisely predicted the strain for the onset of the crease.[4]

Before detailed analysis, using similar scaling analysis for the wrinkles with small wavelength, we expect that the gravitational force will have negligible effects on the onset of creases, because the crease size is the only relevant length scale, which is infinitesimal for incipient creases. The prediction is verified in the following numerical analyses.

To obtain the strain for the onset of creases, following Hong *et al.*,[4] we calculate the free energy difference ΔU between an elastomer with homogenously deformation and

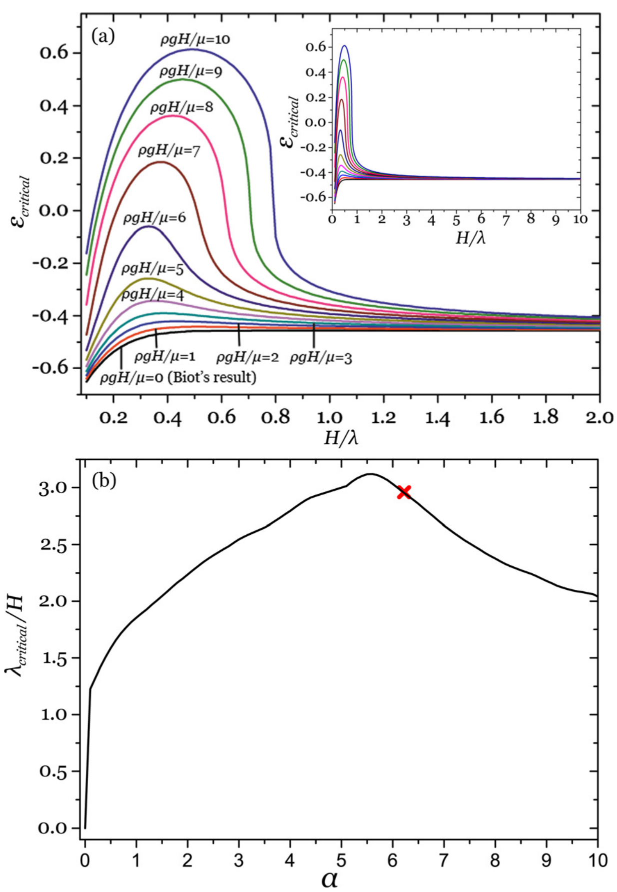

FIG. 2. (a) Critical strains for wrinkles with different wavelength for different gravitational forces (inset plot for a wider range of wavelength). (b) Wavelength of the critical mode of wrinkles as a function of dimensionless gravity. Red Cross point is from Ref. 5.

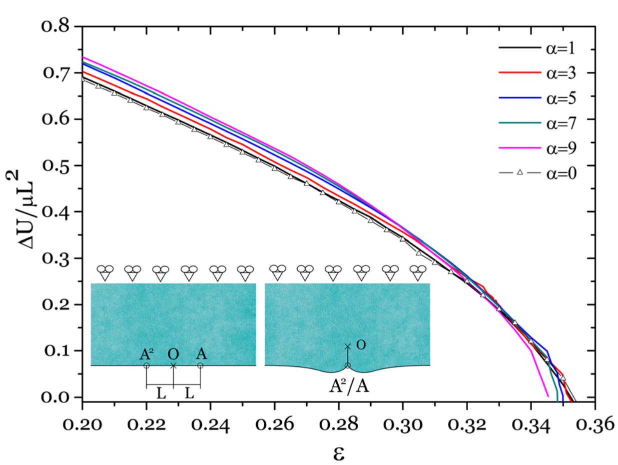

FIG. 3. Free energy differences between an elastomer with the homogenous deformation and the one with crease of infinitesimal depth under gravitational forces. The numerical results show that the effects of gravitational forces on the formation of creases are negligible.

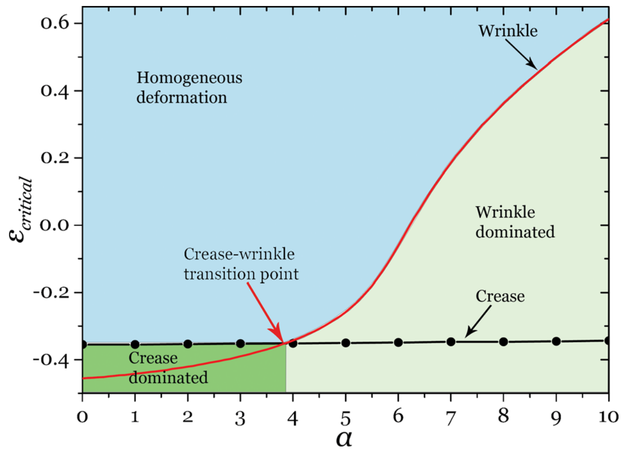

FIG. 4. Comparison of the critical strain of the onset of wrinkles and crease under gravity. The two critical conditions intersect at $\alpha \sim 3.8$, indicating the possible transition between crease and wrinkle state. When $\alpha < 3.8$, crease is the surface instability mode when an elastomer is under compression. For $\alpha > 3.8$, wrinkle is the surface instability mode.

the one with a prescribed crease of small depth L, using FEM simulation (as shown in Fig. 3 inset). The free energy of the elastomer is equal to the summation of the elastic energy and the gravity potential of the elastomer. For an incipient crease, its depth is the only length scale, so

$$\Delta U = \mu L^2 f(\varepsilon, \alpha), \tag{14}$$

where the dimensionless number $f(\varepsilon,\alpha)$ is to be calculated and is a function of applied strain ε and the dimensionless gravity α. If $\Delta U > 0$, the homogeneously deformed elastomer has lower free energy. If $\Delta U < 0$, the crease state has lower free energy. Consequently, the critical condition for the onset of crease is

$$f(\varepsilon, \alpha) = 0. \tag{15}$$

As shown in Fig. 3, which clearly indicates that the effects of gravity on the strain of the onset of creases is negligible.

The critical conditions for the onset of creases and the critical mode of wrinkles are both plotted in Fig. 4. When the strain is larger than the critical strain for both creases and wrinkles, the elastomer will deform homogeneously with keeping its surface flat. The surface of elastomer will form creases or wrinkles, when the strain is smaller than the critical strain for the onset of creases or wrinkles (whichever is larger). Based on the calculation, we can divide Fig. 4 into three domains, which are homogenous deformation, creasing state, and wrinkling state, respectively. When gravity is small ($\alpha < 3.8$) and the compression strain exceeds 35%, creasing instability is formed. When gravity is large ($\alpha > 3.8$), wrinkling instability develops prior to the creasing instability with an increasing critical strain.

In summary, surface instability of a soft elastic solid has been recently intensively studied, when the solid is subject to either compression or gravitational force. In the article, we investigate surface instability of a soft elastic solid under both pre-stretch and gravity using analytical analyses and numerical simulations. We found that the magnitude of gravity may determine the selection of surface instability mode when the soft solid is under compression. When the gravity is small, a compressed surface develops to crease instability. When the gravity is large, wrinkle is formed prior to the crease. The transition between crease instability and wrinkle instability is governed by magnitude of gravity. It has been shown that wrinkling of a uniform elastomer is extremely unstable and very difficult to observe.[13] In the article, we demonstrate that large gravitational force may stabilize the wrinkles on the surface of an elastomer under compression.

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