Constitutive theories based on the multiplicative decomposition of deformation gradient: Thermoelasticity, elastoplasticity, and biomechanics

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Some fundamental issues in the formulation of constitutive theories of material response based on the multiplicative decomposition of the deformation gradient are reviewed, with focus on finite deformation thermoelasticity, elastoplasticity, and biomechanics. The constitutive theory of isotropic thermoelasticity is first considered. The stress response and the entropy expression are derived in the case of quadratic dependence of the elastic strain energy on the finite elastic strain. Basic kinematic and kinetic aspects of the phenomenological and single crystal elastoplasticity within the framework of the multiplicative decomposition are presented. Attention is given to additive decompositions of the stress and strain rates into their elastic and plastic parts. The constitutive analysis of the stress-modulated growth of pseudo-elastic soft tissues is then presented. The elastic and growth parts of the deformation gradient and the rate of deformation tensor are defined and used to construct the corresponding rate-type biomechanic theory. The structure of the evolution equation for growth-induced stretch ratio is discussed. There are 112 references cited in this review article. [DOI: 10.1115/1.1591000]

1 INTRODUCTION

The objective of this survey is to give an overview of the application of the multiplicative decomposition of the deformation gradient in constitutive theories of finite deformation thermoelasticity, elastoplasticity, and biomechanics. The multiplicative decomposition of the deformation gradient is based on an intermediate material configuration, which is obtained by a conceptual destressing of the currently deformed material configuration to zero stress. The significance of such configuration for material modeling was pointed out by Eckart [1], Kröner [2], and Sedov [3], but its formal introduction in nonlinear continuum mechanics can be attributed to Stojanović *et al* [4] in the case of finite deformation thermoelasticity, and to Lee [5] in the case of phenomenological finite deformation elastoplasticity. The decomposition was subsequently extended and used with much success in modeling the elastoplastic deformation of single crystals [6–10]. More recently, following the work of Rodrigez *et al* [11], the multiplicative decomposition of the deformation gradient was applied in biomechanics to study the stress-modulated growth of pseudo-elastic soft tissues [12–15]. A survey of the application of the multiplicative decomposition in these three areas of nonlinear continuum mechanics is presented in this review.

The formulation of the constitutive theory of finite deformation thermoelasticity is first presented. The intermediate configuration is introduced here by a conceptual isothermal destressing of the current material configuration to zero stress. The total deformation gradient is then decomposed into the product of purely elastic and thermal parts. Such an approach was first used by Stojanović *et al* [4,16] in the constitutive study of nonpolar and polar thermoelastic materials. However, in contrast to the decomposition of elastoplastic deformation gradient, discussed below, the decomposition of the thermoelastic deformation gradient received far less attention in the mechanics community. Some revived interest has recently been shown in the work by Miehe [17], Holzapfel and Simo [18], Imam and Johnson [19], and Vujošević and Lubarda [20]. The presentation in Section 2 follows the latter work. The considerations are restricted to elastically and thermally isotropic materials, with an outlined extension to transversely isotropic and orthotropic materials. The stress and entropy expressions are derived in the case of quadratic dependence of the elastic strain energy on the finite elastic strain.

Some fundamental kinematic and kinetic aspects of finite deformation elastoplasticity theory within the framework of the multiplicative decomposition are presented in Section 3. The intermediate configuration is obtained from the deformed material configuration by elastic destressing to zero stress. It differs from the initial configuration by the residual or plastic deformation, and from the current configuration by

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the reversible or elastic deformation. The corresponding decomposition of the elastoplastic deformation gradient into its elastic and plastic part was introduced by Lee [5]. Related early contributions also include Backman [21], Lee and Liu [22], Fox [23], Willis [24], Mandel [25,26], and Kröner and Teodosiu [27]. The decomposition received a great deal of attention in the phenomenological theory of elastoplasticity during past three decades. Representative references include Freund [28], Sidoroff [29], Kleiber [30], Nemat-Nasser [31,32], Lubarda and Lee [33], Johnson and Bammann [34], Simo and Ortiz [35], Needleman [36], Dashner [37], Dafalias [38,39], Agah-Tehrani *et al* [40], Van der Giessen [41], Moran *et al* [42], Naghdi [43], Aravas [44], Lubarda and Shih [45], Xiao *et al* [46], and Lubarda and Benson [47]. The multiplicative decomposition was further extended and successfully applied to model the elastoplastic deformation of single crystals, in which the crystallographic slip is assumed to be the only mechanism of plastic deformation. The plastic part of deformation gradient accounts for the crystallographic slip, while the elastic part accounts for the lattice stretching and rotation; Asaro and Rice [6], Hill and Havner [7], Asaro [8,9], Havner [10], Aravas and Aifantis [48], Bassani [49], Lubarda [50], and Gurtin [51]. The constitutive analysis of single crystal plasticity within the framework of multiplicative decomposition is also presented in Section 3, with an accent given to additive decompositions of the stress and strain rates into their elastic and plastic parts.

The third area of nonlinear continuum mechanics in which the multiplicative decomposition of deformation gradient was applied is biomechanics. The soft tissues, such as blood vessels and tendons, can experience large deformations during their stress-modulated growth. In describing this process, Rodrigez *et al* [11] decomposed the corresponding deformation gradient into its elastic and growth parts. Such decomposition was further utilized by Taber and Eggers [12], Chen and Hoger [13], Klisch and Van Dyke [14], Lubarda and Hoger [15], Taber and Perucchio [52], and Hoger *et al* [53]. In Section 4, we present an analysis of the stress-modulated growth of isotropic pseudo-elastic soft tissues by using this decomposition. The rate-type biomechanic theory is constructed, based on additive decomposition of the rate of deformation into its elastic and growth parts, and an appealing structure of the evolution equation for the growth-induced stretch ratio. The concluding remarks on the multiplicative decomposition of deformation gradient and its role in nonlinear continuum mechanics are given in Section 5.

2 THERMOELASTICITY

In the constitutive theory of thermoelastic material response the intermediate configuration $\mathcal{B}_\theta$ is introduced by isothermal elastic destressing of the current material configuration $\mathcal{B}$ to zero stress (Fig. 1). If the isothermal elastic deformation gradient from $\mathcal{B}_\theta$ to $\mathcal{B}$ is $\mathbf{F}_e$, and the thermal deformation gradient from $\mathcal{B}_o$ to $\mathcal{B}_\theta$ is $\mathbf{F}_\theta$, the total deformation gradient $\mathbf{F}$ can be decomposed as

$$\mathbf{F}=\mathbf{F}_e\cdot\mathbf{F}_\theta \tag{2.1}$$

This decomposition was introduced in finite-strain thermoelasticity by Stojanović and his associates [4,16], and further employed by Stojanović [54], Mićunović [55], and Lu and Pister [56]. For the inhomogeneous deformation and temperature fields, only $\mathbf{F}$ is a true deformation gradient. The mappings from $\mathcal{B}_\theta$ to $\mathcal{B}$ and from $\mathcal{B}_o$ to $\mathcal{B}_\theta$, on the other hand, are generally not continuous one-to-one mappings, so that $\mathbf{F}_e$ and $\mathbf{F}_\theta$ are defined as the point functions or the local deformation gradients. The decomposition (2.1) is not unique because an arbitrary rigid-body rotation can be superposed to $\mathcal{B}_\theta$ preserving it unstressed. However, the gradient $\mathbf{F}_\theta$ can be specified uniquely in each considered case, depending on the type of material anisotropy. For example, for an orthotropic material with the principal axes of orthotropy parallel to unit vectors $\mathbf{m}^o$, $\mathbf{n}^o$, and $\mathbf{m}^o\times\mathbf{n}^o$ in the configuration $\mathcal{B}_o$, the gradient $\mathbf{F}_\theta$ is specified by [57]

$$\mathbf{F}_\theta=\vartheta\,\mathbf{I}+(\beta-\vartheta)\mathbf{m}^o\otimes\mathbf{m}^o+(\gamma-\vartheta)\mathbf{n}^o\otimes\mathbf{n}^o \tag{2.2}$$

The stretch ratios due to thermal expansion in the orthogonal directions $\mathbf{m}^o$ and $\mathbf{n}^o$ are $\beta=\beta(\theta)$ and $\gamma=\gamma(\theta)$, while $\vartheta=\vartheta(\theta)$ is the stretch ratio in the direction $\mathbf{m}^o\times\mathbf{n}^o$. The second-order unit tensor is denoted by $\mathbf{I}$. The modification of the representation (2.2) for transversely isotropic materials is straightforward.

The elastic Lagrangian strain and its rate are

$$\mathbf{E}_e=\mathbf{F}_\theta^{-T}\cdot(\mathbf{E}-\mathbf{E}_\theta)\cdot\mathbf{F}_\theta^{-1} \tag{2.3}$$

$$\dot{\mathbf{E}}_e=\mathbf{F}_\theta^{-T}\cdot\dot{\mathbf{E}}\cdot\mathbf{F}_\theta^{-1}-\mathbf{L}_\theta^s-\mathbf{E}_e\cdot\mathbf{L}_\theta-\mathbf{L}_\theta^T\cdot\mathbf{E}_e \tag{2.4}$$

where $\mathbf{L}_\theta=\dot{\mathbf{F}}_\theta\cdot\mathbf{F}_\theta^{-1}$ is the velocity gradient in the intermediate configuration, and $\mathbf{L}_\theta^s=(\mathbf{L}_\theta+\mathbf{L}_\theta^T)/2$ stands for its symmetric part. The elastic and thermal strains are defined by

$$\mathbf{E}_e=\frac{1}{2}(\mathbf{F}_e^T\cdot\mathbf{F}_e-\mathbf{I}),\quad \mathbf{E}_\theta=\frac{1}{2}(\mathbf{F}_\theta^T\cdot\mathbf{F}_\theta-\mathbf{I}) \tag{2.5}$$

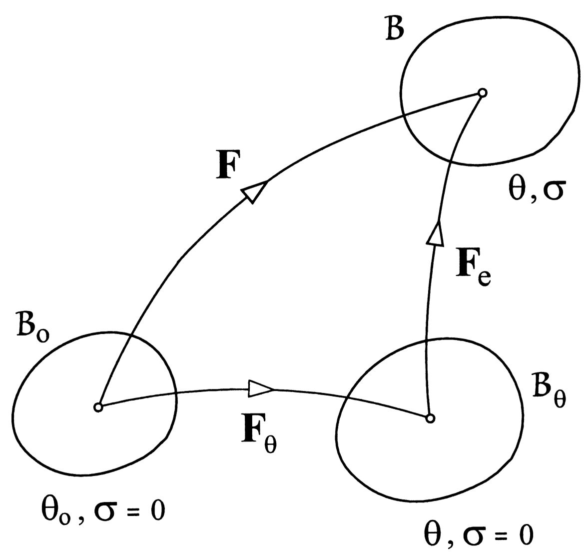

Fig. 1 The intermediate configuration $\mathcal{B}_\theta$ at a nonuniform temperature θ is obtained from the deformed configuration $\mathcal{B}$ by isothermal destressing to zero stress. The deformation gradient from initial to deformed configuration $\mathbf{F}$ is decomposed into elastic part $\mathbf{F}_e$ and thermal part $\mathbf{F}_\theta$, such that $\mathbf{F}=\mathbf{F}_e\cdot\mathbf{F}_\theta$.

The subsequent analysis will be restricted to isotropic materials, for which the thermal part of the deformation gradient is

$$\mathbf{F}_\theta = \vartheta(\theta)\mathbf{I} \tag{2.6}$$

The scalar $\vartheta = \vartheta(\theta)$ is the thermal stretch ratio in any material direction. In this case, the elastic and thermal strains become

$$\mathbf{E}_e = \frac{1}{\vartheta^2}(\mathbf{E} - \mathbf{E}_\theta), \quad \mathbf{E}_\theta = \frac{1}{2}(\vartheta^2 - 1)\mathbf{I} \tag{2.7}$$

The relationship holds

$$\mathbf{I} + 2\mathbf{E} = \vartheta^2(\mathbf{I} + 2\mathbf{E}_e) \tag{2.8}$$

Since the thermal stretch ratio ϑ and the coefficient of thermal expansion α are related by

$$\alpha(\theta) = \frac{1}{\vartheta}\frac{d\vartheta}{d\theta} \tag{2.9}$$

the rate of elastic strain can be written as

$$\dot{\mathbf{E}}_e = \frac{1}{\vartheta^2(\theta)}[\dot{\mathbf{E}} - \alpha(\theta)(\mathbf{I} + 2\mathbf{E})\dot{\theta}] \tag{2.10}$$

2.1 Stress response

Within the model of the multiplicative decomposition, the Helmholtz free energy can be conveniently split into two parts, such that

$$\psi = \psi_e(\mathbf{E}_e, \theta) + \psi_\theta(\theta) \tag{2.11}$$

where ψ_e is an isotropic function of the elastic strain $\mathbf{E}_e$ and the temperature θ. This decomposition is physically appealing because the function $\psi_e(\mathbf{E}_e, \theta)$ can be taken as one of the well-known strain energy functions of the isothermal finite-strain elasticity, except that the coefficients of the strain-dependent terms are the functions of temperature, while the function $\psi_\theta(\theta)$ can be separately adjusted in accord with experimental data for the specific heat. Other representations of ψ are possible, such as those used by Lu and Pister [56], and Imam and Johnson [19]. The time-rate of the free energy in Eq. (2.11) is

$$\dot{\psi} = \frac{\partial \psi_e}{\partial \mathbf{E}_e} : \dot{\mathbf{E}}_e + \frac{\partial \psi_e}{\partial \theta}\dot{\theta} + \frac{d\psi_\theta}{d\theta}\dot{\theta} \tag{2.12}$$

Upon the substitution of Eq. (2.10), there follows

$$\dot{\psi} = \frac{1}{\vartheta^2}\frac{\partial \psi_e}{\partial \mathbf{E}_e} : \dot{\mathbf{E}} - \left[\frac{\alpha}{\vartheta^2}\frac{\partial \psi_e}{\partial \mathbf{E}_e} : (\mathbf{I} + 2\mathbf{E}) - \frac{\partial \psi_e}{\partial \theta} - \frac{d\psi_\theta}{d\theta}\right]\dot{\theta} \tag{2.13}$$

The comparison with the energy equation,

$$\dot{\psi} = \frac{1}{\varrho_o}\mathbf{T} : \dot{\mathbf{E}} - \eta\dot{\theta} \tag{2.14}$$

establishes the constitutive relations for the symmetric Piola–Kirchhoff stress $\mathbf{T}$ and the specific entropy η. These are

$$\mathbf{T} = \frac{\varrho_o}{\vartheta^2}\frac{\partial \psi_e}{\partial \mathbf{E}_e} \tag{2.15}$$

$$\eta = \alpha\frac{\partial \psi_e}{\partial \mathbf{E}_e} : (\mathbf{I} + 2\mathbf{E}_e) - \frac{\partial \psi_e}{\partial \theta} - \frac{d\psi_\theta}{d\theta} \tag{2.16}$$

In view of the relationship $\varrho_o = \vartheta^3 \varrho_\theta$, between the densities ϱ_o in the configuration $\mathcal{B}_o$ and ϱ_θ in the configuration $\mathcal{B}_\theta$, the stress response in Eq. (2.15) can also be written as

$$\mathbf{T} = \vartheta \mathbf{T}_e, \quad \mathbf{T}_e = \varrho_\theta \frac{\partial \psi_e}{\partial \mathbf{E}_e} \tag{2.17}$$

For example, suppose that ψ_e is a quadratic function of the elastic strain components, such that

$$\varrho_\theta \psi_e = \frac{1}{2}\lambda(\theta)(\mathrm{tr}\mathbf{E}_e)^2 + \mu(\theta)\mathbf{E}_e : \mathbf{E}_e \tag{2.18}$$

where $\lambda(\theta)$ and $\mu(\theta)$ are the temperature-dependent Lamé moduli. It follows that

$$\mathbf{T}_e = \boldsymbol{\Lambda}_e(\theta) : \mathbf{E}_e, \quad \boldsymbol{\Lambda}_e(\theta) = \lambda(\theta)\mathbf{I} \otimes \mathbf{I} + 2\mu(\theta)\mathbf{II} \tag{2.19}$$

The temperature-dependent elastic moduli tensor is $\boldsymbol{\Lambda}_e(\theta)$, while $\mathbf{II}$ stands for the fourth-order unit tensor with rectangular components

$$II_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.20}$$

The rectangular components of the second-order unit tensor are the Kronecker deltas δ_{ij}. Consequently, by substituting Eqs. (2.10) and (2.19) into $\mathbf{T} = \vartheta \mathbf{T}_e$, the stress response becomes

$$\mathbf{T} = \frac{1}{\vartheta(\theta)}[\lambda(\theta)(\mathrm{tr}\,\mathbf{E})\mathbf{I} + 2\mu(\theta)\mathbf{E}] - \frac{3}{2}\left[\vartheta(\theta) - \frac{1}{\vartheta(\theta)}\right]\kappa(\theta)\mathbf{I} \tag{2.21}$$

The temperature-dependent bulk modulus is $\kappa(\theta)$. This is an exact expression for the thermoelastic stress response in the case of quadratic representation of ψ_e in terms of the finite elastic strain $\mathbf{E}_e$. If the Lamé moduli are taken to be temperature-independent, and if the approximation $\vartheta(\theta) \approx 1 + \alpha_o(\theta - \theta_o)$ is used (α_o being the coefficient of linear thermal expansion at $\theta = \theta_o$), Eq. (2.21) reduces to

$$\mathbf{T} = \lambda_o(\mathrm{tr}\mathbf{E})\mathbf{I} + 2\mu_o\mathbf{E} - 3\alpha_o(\theta - \theta_o)\kappa_o\mathbf{I} \tag{2.22}$$

When $\mathbf{E}$ and $\mathbf{T}$ are interpreted as the infinitesimal strain and the Cauchy stress, this equation coincides with the well-known Duhamel–Neumann expression of isotropic linear thermoelasticity (eg, Carlson [58] and Nowacki [59]).

2.2 Entropy expression

In the case of quadratic strain energy representation (2.18), there is a relationship $\varrho_o \psi_e = \vartheta^3 \mathbf{T}_e : \mathbf{E}_e/2$, so that

$$\varrho_o\left(\frac{\partial \psi_e}{\partial \theta}\right)_{\mathbf{E}_e} = \frac{3}{2}\vartheta^2\frac{d\vartheta}{d\theta}\mathbf{T}_e : \mathbf{E}_e + \frac{1}{2}\vartheta^3\left(\frac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} : \mathbf{E}_e \tag{2.23}$$

Alternatively, by using Eq. (2.8), this can be recast as

$$\varrho_{\mathrm{o}}\left(\frac{\partial \psi_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}} = \frac{3}{2}\alpha\left[\mathbf{T}:\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\operatorname{tr}\mathbf{T}\right] + \frac{1}{2}\vartheta^3\left(\frac{\partial \mathbf{T}_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}}:\mathbf{E}_{\mathrm{e}} \tag{2.24}$$

The coefficient of thermal expansion α is defined by Eq. (2.9). It is readily verified that

$$\vartheta\left(\frac{\partial \mathbf{T}_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}} = \left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} + \alpha(\mathbf{T} + 3\vartheta\kappa\mathbf{I}) \tag{2.25}$$

and

$$\vartheta^3\left(\frac{\partial \mathbf{T}_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}}:\mathbf{E}_{\mathrm{e}} = \left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}}:\left[\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\,\mathbf{I}\right] + \alpha\left[\mathbf{T}:\mathbf{E} + \frac{1}{2}(1 + \vartheta^2)\operatorname{tr}\mathbf{T}\right] \tag{2.26}$$

Inserting Eq. (2.26) into Eq. (2.24) gives

$$\varrho_{\mathrm{o}}\left(\frac{\partial \psi_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}} = 2\,\alpha\,\mathbf{T}:\mathbf{E} + \frac{1}{2}\alpha(2 - \vartheta^2)\operatorname{tr}\mathbf{T} + \frac{1}{2}\left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}}:\left[\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\mathbf{I}\right] \tag{2.27}$$

When this is substituted into Eq. (2.16), the entropy becomes

$$\eta = \frac{1}{2\varrho_{\mathrm{o}}}\left[3\vartheta\alpha\kappa\,\mathbf{I} - \left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}}\right]:\left[\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\mathbf{I}\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta} \tag{2.28}$$

Recalling the standard expression for the latent heat ℓ_E, we finally have

$$\eta = \frac{1}{2}\left(\frac{1}{\theta}\ell_E + \frac{3}{\varrho_{\mathrm{o}}}\vartheta\,\alpha\,\kappa\,\mathbf{I}\right):\left[\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\,\mathbf{I}\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta} \tag{2.29}$$

This is an exact expression for the entropy η within the approximation used for the elastic strain energy function. The second-order tensor of the latent heat ℓ_E can be calculated from Eq. (2.25) as

$$\ell_E = -\frac{1}{\varrho_{\mathrm{o}}}\theta\left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} = -\frac{1}{\varrho_{\mathrm{o}}}\theta\left[\vartheta\left(\frac{\partial \mathbf{T}_{\mathrm{e}}}{\partial \theta}\right)_{\mathbf{E}_{\mathrm{e}}} - \alpha(\mathbf{T} + 3\,\vartheta\,\kappa\,\mathbf{I})\right] \tag{2.30}$$

which gives

$$\ell_E = \frac{1}{\varrho_{\mathrm{o}}}\theta\left\{\alpha\,(\mathbf{T} + 3\,\vartheta\,\kappa\,\mathbf{I}) - \frac{1}{\vartheta}\frac{\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{e}}}{\mathrm{d}\theta}:\left[\mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\mathbf{I}\right]\right\} \tag{2.31}$$

If the elastic moduli are independent of the temperature, and if the stress components are much smaller than the elastic bulk modulus, then the specific heat becomes $\ell_E = 3\vartheta\alpha\theta\kappa\mathbf{I}/\varrho_{\mathrm{o}}$, while the entropy expression (2.29) reduces to

$$\eta = \frac{3}{\varrho_{\mathrm{o}}}\vartheta\,\alpha\,\kappa\left[\operatorname{tr}\mathbf{E} - \frac{3}{2}(\vartheta^2 - 1)\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta} \tag{2.32}$$

The function ψ_θ can be selected according to experimental data for the specific heat $c_E = \theta\partial\eta/\partial\theta$. For example, if we take

$$\psi_\theta = -\frac{1}{2}\left(\frac{c_E^{\mathrm{o}}}{\theta_{\mathrm{o}}} + \frac{9}{\varrho_{\mathrm{o}}}\alpha_{\mathrm{o}}^2\,\kappa_{\mathrm{o}}\right)(\theta - \theta_{\mathrm{o}})^2 \tag{2.33}$$

then Eq. (2.32) becomes

$$\eta = \frac{3}{\varrho_{\mathrm{o}}}\alpha_{\mathrm{o}}\,\kappa_{\mathrm{o}}\operatorname{tr}\mathbf{E} + \frac{c_E^{\mathrm{o}}}{\theta_{\mathrm{o}}}(\theta - \theta_{\mathrm{o}}) \tag{2.34}$$

which is in agreement with the classical result from the linearized theory of thermoelasticity [58]. The approximations $\alpha \approx \alpha_0$ and $\vartheta(\theta) \approx 1 + \alpha_{\mathrm{o}}(\theta - \theta_{\mathrm{o}})$ are used in the above derivation.

3 ELASTOPLASTICITY

The intermediate configuration in finite-deformation elastoplasticity is obtained from the current configuration by elastic destressing to zero stress (Fig. 2). It differs from the initial configuration by a residual or plastic deformation and from the current configuration by a reversible or elastic deformation. The corresponding multiplicative decomposition of elastoplastic deformation gradient into its elastic and plastic part was introduced by Lee [5] as

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{p}} \tag{3.1}$$

In the case when elastic destressing to zero stress is not physically achievable due to possible onset of the reverse plastic deformation before the state of zero stress is reached, the intermediate configuration can be conceptually introduced by a virtual destressing to zero stress, locking all inelastic structural changes that would occur during the actual destressing. The deformation gradients $\mathbf{F}_{\mathrm{e}}$ and $\mathbf{F}_{\mathrm{p}}$ are not uniquely defined because the intermediate configuration is not unique; arbitrary local material rotations can be superposed to intermediate configuration preserving it unstressed. This has been extensively discussed in the literature by Green and Naghdi [60], Lubarda and Lee [33], Casey and Naghdi [61], Kleiber and Raniecki [62], Dashner [37], Casey [63], Dafalias [38], Lubarda [64], and others. In the applications, however, the decomposition can be made unique by additional specifications dictated by the nature of the considered material model. For example, for elastically isotropic materials which remain isotropic in the course of deformation the stress response from $\mathcal{B}_{\mathrm{p}}$ to $\mathcal{B}$ does not depend on the rotation $\mathbf{R}_{\mathrm{e}}$ appearing in the polar decomposition of elastic deformation gradient $\mathbf{F}_{\mathrm{e}} = \mathbf{V}_{\mathrm{e}} \cdot \mathbf{R}_{\mathrm{e}}$. Consequently, the intermediate configuration in this case can be defined uniquely by requiring that elastic unloading takes place without rotation. Other choices are possible and are discussed in [64,65].

In contrast to finite-strain thermoelasticity, considered in the previous section, the elastoplasticity is a path-dependent deformation process, which is commonly analyzed by an incremental procedure, following the prescribed loading or deformation history. This requires the use of the rate-type measures of deformation. By introducing the multiplicative decomposition of deformation gradient (3.1), the velocity gradient in the current configuration $\mathbf{L}=\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}$ becomes

$$\mathbf{L}=\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1}+\mathbf{F}_e\cdot(\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1})\cdot\mathbf{F}_e^{-1} \tag{3.2}$$

The rate of deformation $\mathbf{D}$ and the spin $\mathbf{W}$ are given by the symmetric and antisymmetric part of $\mathbf{L}$, so that

$$\mathbf{D}=(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_s+[\mathbf{F}_e\cdot(\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1})\cdot\mathbf{F}_e^{-1}]_s \tag{3.3}$$

$$\mathbf{W}=(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_a+[\mathbf{F}_e\cdot(\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1})\cdot\mathbf{F}_e^{-1}]_a \tag{3.4}$$

For the later purposes, the second spin tensor appearing on the right-hand side of Eq. (3.4) is conveniently denoted by

$$\boldsymbol{\omega}_p=[\mathbf{F}_e\cdot(\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1})\cdot\mathbf{F}_e^{-1}]_a \tag{3.5}$$

3.1 Partition of elastoplastic rate of deformation

Large plastic deformations can affect elastic properties of the material and change its elastic symmetry group. This, for example, can happen due to grain (lattice) rotations in a polycrystalline metal sample and resulting crystallographic texture. In such cases, the damage variables (scalars, vectors, second- or higher-order tensors) can be introduced to describe the degradation of elastic properties and their directional changes caused by plastic deformation [66–68]. On the other hand, in the range of small or moderately large deformations, it may be appropriate to assume that plastic deformation does not affect elastic properties of the material. In this case, the elastic response of an isotropic material is independent of the rotation superposed to intermediate configuration, and is given by

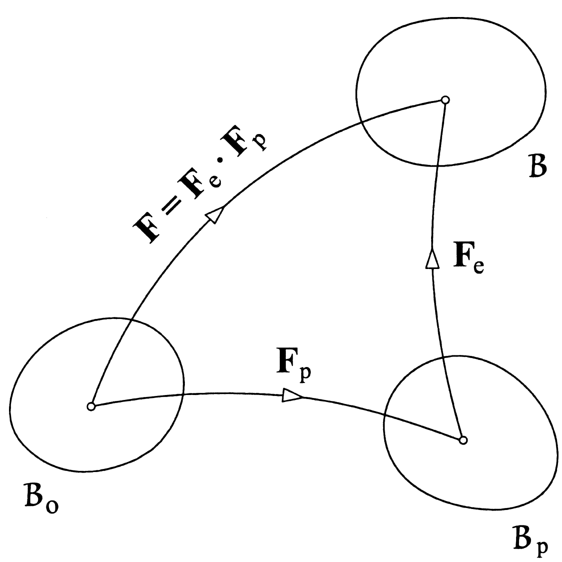

Fig. 2 The intermediate configuration $\mathcal{B}_p$ is obtained from the deformed configuration $\mathcal{B}$ by destressing to zero stress. The elastoplastic deformation gradient is decomposed into its elastic and plastic part, such that $\mathbf{F}=\mathbf{F}_e\cdot\mathbf{F}_p$.

$$\boldsymbol{\tau}=\mathbf{F}_e\cdot\frac{\partial\Psi_e(\mathbf{E}_e)}{\partial\mathbf{E}_e}\cdot\mathbf{F}_e^T \tag{3.6}$$

The elastic strain energy per unit unstressed volume ($\Psi_e=\varrho_o\psi_e$) is here an isotropic function of the Lagrangian strain $\mathbf{E}_e$. The plastic deformation is assumed to be incompressible ($\det\mathbf{F}_e=\det\mathbf{F}$), so that $\boldsymbol{\tau}=(\det\mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress (the Cauchy stress $\boldsymbol{\sigma}$ weighted by $\det\mathbf{F}$). By differentiating Eq. (3.6), we obtain

$$\dot{\boldsymbol{\tau}}-(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})^T=\bar{\boldsymbol{\mathcal{L}}}_e:(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_s \tag{3.7}$$

The rectangular components of the fourth-order elastic moduli tensor $\bar{\boldsymbol{\mathcal{L}}}_e$ are

$$\bar{\mathcal{L}}^e_{ijkl}=F^e_{im}F^e_{jn}\frac{\partial^2\Psi_e}{\partial E^e_{mn}\partial E^e_{pq}}F^e_{kp}F^e_{lq} \tag{3.8}$$

Equation (3.7) can be rewritten as

$$\dot{\boldsymbol{\tau}}-(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_a\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_a=\boldsymbol{\mathcal{L}}_e:(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_s \tag{3.9}$$

with the modified instantaneous moduli given by

$$\mathcal{L}^e_{ijkl}=\bar{\mathcal{L}}^e_{ijkl}+\frac{1}{2}(\tau_{ik}\delta_{jl}+\tau_{jk}\delta_{il}+\tau_{il}\delta_{jk}+\tau_{jl}\delta_{ik}) \tag{3.10}$$

The elastic deformation gradient $\mathbf{F}_e$ is defined relative to intermediate configuration which evolves during elastoplastic deformation. This causes two difficulties in the identification of the elastic part of the rate of deformation [45]. First, since $\mathbf{F}_e$ and $\mathbf{F}_p$ are specified only to within an arbitrary rotation, the velocity gradient $\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1}$ and its symmetric and antisymmetric parts are not unique. Second, the deforming intermediate configuration also contributes to elastic rate of deformation, which is not in general given only by $(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_s$. To overcome these difficulties, a kinetic definition of elastic strain increment is adopted according to which $\mathbf{D}_e\,dt$ is defined as a reversible part of the total strain increment $\mathbf{D}\,dt$, recovered upon loading-unloading cycle of the Jaumann stress increment $\overset{\circ}{\boldsymbol{\tau}}dt$. Thus, if $\boldsymbol{\mathcal{L}}_e^{-1}$ designates the instantaneous elastic compliances tensor, the inverse of the instantaneous elastic moduli tensor (3.10), we require that

$$\mathbf{D}_e=\boldsymbol{\mathcal{L}}_e^{-1}:\overset{\circ}{\boldsymbol{\tau}},\qquad \overset{\circ}{\boldsymbol{\tau}}=\dot{\boldsymbol{\tau}}-\mathbf{W}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{W} \tag{3.11}$$

The remaining part of the total rate of deformation,

$$\mathbf{D}_p=\mathbf{D}-\mathbf{D}_e \tag{3.12}$$

is the plastic part, which gives a residual strain increment left upon the considered infinitesimal cycle of stress. If the material obeys the Ilyushin [69] postulate of positive net work in an isothermal cycle of strain that involves plastic deformation, the so defined plastic rate of deformation is codirectional with the outward normal to a locally smooth yield surface in the Cauchy stress space. This definition of plastic rate of deformation was introduced in the constitutive analysis of elastoplastic deformation by Hill and Rice [70] and Hill [71].

To identify in Eq. (3.9) the elastic strain rate, in accord with the kinetic definition (3.11), we eliminate $(\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1})_a$ in terms of $\mathbf{W}$ and $\boldsymbol{\omega}_p$, to obtain

$$\overset{\circ}{\boldsymbol{\tau}}=\boldsymbol{\mathcal{L}}_{\mathrm{e}}:(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}-\boldsymbol{\omega}_{\mathrm{p}}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\boldsymbol{\omega}_{\mathrm{p}} \tag{3.13}$$

Consequently, the elastic rate of deformation is given by

$$\mathbf{D}_{\mathrm{e}}=(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}-\boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1}:(\boldsymbol{\omega}_{\mathrm{p}}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_{\mathrm{p}}) \tag{3.14}$$

The associated plastic rate of deformation is

$$\mathbf{D}_{\mathrm{p}}=[\mathbf{F}_{\mathrm{e}}\cdot(\dot{\mathbf{F}}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{p}}^{-1})\cdot\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}}+\boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1}:(\boldsymbol{\omega}_{\mathrm{p}}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_{\mathrm{p}}) \tag{3.15}$$

Since $\boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1}$ and $\overset{\circ}{\boldsymbol{\tau}}$ in Eq. (3.11) are independent of the superposed rotation to intermediate configuration, Eq. (3.14) specifies $\mathbf{D}_{\mathrm{e}}$ uniquely. In contrast, its constituents, $(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}$ and the term dependent on the spin $\boldsymbol{\omega}_{\mathrm{p}}$, do depend on the choice of intermediate configuration. Similar remarks apply to plastic rate of deformation $\mathbf{D}_{\mathrm{p}}$ in its representation (3.15). It is $\mathbf{D}_{\mathrm{p}}$ that is normal to the yield surface, and not the first term on the right-hand side of Eq. (3.15). In transforming the velocity gradient $\dot{\mathbf{F}}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{p}}^{-1}$ from intermediate to current configuration by elastic deformation, the corresponding rate of deformation $[\mathbf{F}_{\mathrm{e}}\cdot(\dot{\mathbf{F}}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{p}}^{-1})\cdot\mathbf{F}_{\mathrm{e}}^{-1}]_{\mathrm{s}}$ is equal to plastic rate of deformation $\mathbf{D}_{\mathrm{p}}$, with an elastic contribution due to stress rate $\boldsymbol{\omega}_{\mathrm{p}}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_{\mathrm{p}}$ subtracted off; Hill and Havner [7].

If elastic components of strain are infinitesimally small, then the instantaneous elastic compliances tensor is obtained by an explicit inversion of the elastic moduli tensor as

$$\boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1}=\frac{1}{2\mu}\left(\mathbf{II}-\frac{\lambda}{2\mu+3\lambda}\mathbf{I}\otimes\mathbf{I}\right)=\frac{1}{2\mu}\boldsymbol{J}+\frac{1}{3\kappa}\boldsymbol{K} \tag{3.16}$$

where $\boldsymbol{J}+\boldsymbol{K}=\mathbf{II}$, and

$$II_{ijkl}=\frac{1}{2}(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}),\quad K_{ijkl}=\frac{1}{3}\delta_{ij}\,\delta_{kl} \tag{3.17}$$

The right hand side of (3.14) is the correct expression for the elastic rate of deformation, and not $(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}$ alone. Only if the intermediate configuration (*ie*, the rotation $\mathbf{R}_{\mathrm{e}}$ during the destressing program) is chosen such that the spin $\boldsymbol{\omega}_{\mathrm{p}}=\mathbf{0}$, the rate of deformation $(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}$ is exactly equal to $\mathbf{D}_{\mathrm{e}}$. Within the framework under discussion, this choice of the spin represents a geometric (kinematic) specification of the intermediate configuration. It is not a constitutive assumption and has no consequences on (3.14). We could just as well define the intermediate configuration by requiring that the spin $(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{a}}$ vanishes identically. In this case, $\boldsymbol{\omega}_{\mathrm{p}}=\mathbf{W}$. The end result is still equation (3.14), as can be checked by inspection.

The constitutive structure for the plastic part of the rate of deformation tensor is constructed by using the concept of the yield surface. This gives

$$\mathbf{D}_{\mathrm{p}}=\frac{1}{H}\left(\frac{\partial f}{\partial\boldsymbol{\sigma}}\otimes\frac{\partial f}{\partial\boldsymbol{\sigma}}\right):\overset{\circ}{\boldsymbol{\tau}} \tag{3.18}$$

where H is a scalar parameter of the deformation history, and $f=0$ defines the yield surface. For example, in then case of kinematic hardening with the von Mises type yield condition

$$f=\frac{1}{2}(\boldsymbol{\sigma}'-\boldsymbol{\alpha}):(\boldsymbol{\sigma}'-\boldsymbol{\alpha})-K^2=0 \tag{3.19}$$

and the Armstrong–Frederick evolution of the back stress

$$\overset{\circ}{\boldsymbol{\alpha}}=2h\,\mathbf{D}^{\mathrm{p}}-c\,\boldsymbol{\alpha}(\mathbf{D}^{\mathrm{p}}:\mathbf{D}^{\mathrm{p}})^{1/2} \tag{3.20}$$

it follows that

$$\mathbf{D}^{\mathrm{p}}=\frac{1}{2h(1-m)}\frac{(\boldsymbol{\sigma}'-\boldsymbol{\alpha})\otimes(\boldsymbol{\sigma}'-\boldsymbol{\alpha})}{(\boldsymbol{\sigma}'-\boldsymbol{\alpha}):(\boldsymbol{\sigma}'-\boldsymbol{\alpha})}:\overset{\circ}{\boldsymbol{\tau}} \tag{3.21}$$

where

$$m=\frac{c}{2\,h}\frac{(\boldsymbol{\sigma}'-\boldsymbol{\alpha}):\boldsymbol{\alpha}}{[(\boldsymbol{\sigma}'-\boldsymbol{\alpha}):(\boldsymbol{\sigma}'-\boldsymbol{\alpha})]^{1/2}} \tag{3.22}$$

The parameters h and c are the material parameters. Other hardening models are discussed in the books by Khan and Huang [72] and Simo and Hughes [73]. The formulation of the elastoplastic constitutive theory by using the yield surface in strain space is presented by Hill [71], Casey and Naghdi [74], Naghdi [43], and Lubarda [75,76]. Additional references are available in Naghdi's review [43].

The partition of the total rate of deformation into its elastic and plastic parts within the framework of the multiplicative decomposition has been a topic of active research and some debate for number of years. Representative references include Kratochvil [77], Nemat-Nasser [31,32], Lubarda and Lee [33], Johnson and Bammann [34], Simo and Ortiz [35], Needleman [36], Moran *et al* [42], Agah-Tehrani *et al* [40], Dafalias [38,39], Van der Giessen [41], Naghdi [43], Lubarda [64,78], and Xiao *et al* [46]. For elastically anisotropic materials, the papers by Aravas [44], Lubarda [79], and Steinmann *et al* [80] can be consulted.

3.2 Analysis of elastic rate of deformation

The elastic rate of deformation of an elastically isotropic material can be expressed in terms of the kinematic quantities only, as

$$\mathbf{D}_{\mathrm{e}}=(\overset{\circ}{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}=(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}+(\mathbf{F}_{\mathrm{e}}\cdot\boldsymbol{\Omega}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} \tag{3.23}$$

The Jaumann derivative of $\mathbf{F}_{\mathrm{e}}$ is here defined by

$$\overset{\circ}{\mathbf{F}}_{\mathrm{e}}=\dot{\mathbf{F}}_{\mathrm{e}}-\boldsymbol{\Omega}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{e}}+\mathbf{F}_{\mathrm{e}}\cdot\boldsymbol{\Omega}_{\mathrm{p}} \tag{3.24}$$

which represents the rate of $\mathbf{F}_{\mathrm{e}}$ observed in the coordinate systems that rotate with the spin $\boldsymbol{\Omega}_{\mathrm{p}}$ in both the current and the intermediate configuration. The spin $\boldsymbol{\Omega}_{\mathrm{p}}$ is defined as the solution of the matrix equation

$$(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{a}}+(\mathbf{F}_{\mathrm{e}}\cdot\boldsymbol{\Omega}_{\mathrm{p}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{a}}=\mathbf{W} \tag{3.25}$$

The proof for the representation (3.23) proceeds by applying the Jaumann derivative with respect to $\boldsymbol{\Omega}_{\mathrm{p}}$ to both sides of Eq. (3.6), which gives

$$\overset{\circ}{\boldsymbol{\tau}}=(\overset{\circ}{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot(\overset{\circ}{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})^T+\mathbf{F}_{\mathrm{e}}\cdot\left(\frac{\partial^2\Psi_{\mathrm{e}}}{\partial\mathbf{E}_{\mathrm{e}}\otimes\partial\mathbf{E}_{\mathrm{e}}}:\overset{\circ}{\mathbf{E}}_{\mathrm{e}}\right)\cdot\mathbf{F}_{\mathrm{e}}^T \tag{3.26}$$

Since

$$\overset{\circ}{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1}=\mathbf{D}_{\mathrm{e}}+\mathbf{W}-\boldsymbol{\Omega}_{\mathrm{p}} \tag{3.27}$$

the substitution into Eq. (3.26) yields

$$\overset{\circ}{\boldsymbol{\tau}}=\boldsymbol{\mathcal{L}}_{\mathrm{e}}:\mathbf{D}_{\mathrm{e}},\quad \mathbf{D}_{\mathrm{e}}=(\overset{\circ}{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} \tag{3.28}$$

The two contributions to the elastic rate of deformation $\mathbf{D}_e$ in Eq. (3.23) depend on the choice of intermediate configuration, ie, on the elastic rotation $\mathbf{R}_e$ of destressing program, but their sum giving $\mathbf{D}_e$ does not. If elastic destressing is performed without rotation ($\mathbf{R}_e=\mathbf{I}$), the spin $\mathbf{\Omega}_p=\mathbf{\Omega}_p^o$ is the solution of

$$(\dot{\mathbf{V}}_e\cdot\mathbf{V}_e^{-1})_a+(\mathbf{V}_e\cdot\mathbf{\Omega}_p^o\cdot\mathbf{V}_e^{-1})_a=\mathbf{W} \tag{3.29}$$

This uniquely defines $\mathbf{\Omega}_p^o$ in terms of $\mathbf{W}$, $\mathbf{V}_e$ and $\dot{\mathbf{V}}_e$. The elastic rate of deformation (3.23) is in this case

$$\mathbf{D}_e=(\dot{\mathbf{V}}_e\cdot\mathbf{V}_e^{-1})_s=(\dot{\mathbf{V}}_e\cdot\mathbf{V}_e^{-1})_s+(\mathbf{V}_e\cdot\mathbf{\Omega}_p^o\cdot\mathbf{V}_e^{-1})_s \tag{3.30}$$

The first term on the far right-hand side represents the contribution to $\mathbf{D}_e$ from the elastic stretching rate $(\dot{\mathbf{V}}_e\cdot\mathbf{V}_e^{-1})_s$, while the second term depends on the spin $\mathbf{\Omega}_p^o$ and accounts for the effects of deforming and rotating intermediate configuration [31,44,62].

The representation of the elastic rate of deformation in Eq. (3.23) involves only kinematic quantities ($\mathbf{F}_e$ and $\mathbf{\Omega}_p$), while the representation (3.14) involves both kinematic and kinetic quantities. Clearly,

$$(\mathbf{F}_e\cdot\mathbf{\Omega}_p\cdot\mathbf{F}_e^{-1})_s=-\boldsymbol{\mathcal{L}}_e^{-1}:(\boldsymbol{\omega}_p\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_p) \tag{3.31}$$

Note also that the elastic strain expression (3.23) can be recast in the form

$$\mathbf{D}_e=\frac{1}{2}\mathbf{F}_e^{-T}\cdot\overset{\circ}{\mathbf{C}}_e\cdot\mathbf{F}_e^{-1},\quad \overset{\circ}{\mathbf{C}}_e=\dot{\mathbf{C}}_e-\mathbf{\Omega}_p\cdot\mathbf{C}_e+\mathbf{C}_e\cdot\mathbf{\Omega}_p \tag{3.32}$$

The expressions (3.23) and (3.32) hold for elastoplastic deformations of elastically isotropic materials, regardless of whether the material hardens isotropically or anisotropically during the deformation process.

Additional analysis of the elastic rate of deformation and the partition of the total rate of deformation into its elastic and plastic parts can be found in cited papers. There has also been an extensive research devoted to plastic spin and its role in phenomenological elastoplasticity theory. The papers by Lee *et al* [81], Loret [82], Dafalias [83,84], Zbib and Aifantis [85], Van der Giessen [86], Nemat-Nasser [87], Lubarda and Shih [45], and the review by Dafalias [88] can be consulted in this regard.

3.3 Crystal plasticity

In single crystals for which crystallographic slip is assumed to be the only mechanism of plastic deformation, the material flows through the lattice via dislocation motion, while the lattice itself, with the material embedded to it, undergoes elastic deformation and rotation. If the discrete dislocation substructure is ignored, the plastic deformation can be considered to occur in the form of smooth shearing on the slip planes and in the slip directions. This continuum slip model from the pioneering work of Taylor [89] was employed and further developed by Hill and Rice [90], Mandel [91], Asaro and Rice [6], Hill and Havner [7], and Asaro [8,9]. The deformation gradient is decomposed as

$$\mathbf{F}=\mathbf{F}_*\cdot\mathbf{F}_p \tag{3.33}$$

where $\mathbf{F}_p$ is the part due to slip only, while $\mathbf{F}_*$ is due to lattice stretching and rotation (Fig. 3). Denote the unit vector in the slip direction by $\mathbf{s}_o^\alpha$ and the unit normal to the corresponding slip plane in the undeformed configuration by $\mathbf{m}_o^\alpha$, where α designates the slip system. The vector $\mathbf{s}_o^\alpha$ is embedded in the lattice, so that it becomes $\mathbf{s}^\alpha=\mathbf{F}_*\cdot\mathbf{s}_o^\alpha$ in the deformed configuration. The normal to the slip plane in the deformed configuration is defined by the reciprocal vector $\mathbf{m}^\alpha=\mathbf{m}_o^\alpha\cdot\mathbf{F}_*^{-1}$, ie,

$$\mathbf{s}^\alpha=\mathbf{F}_*\cdot\mathbf{s}_o^\alpha,\quad \mathbf{m}^\alpha=\mathbf{m}_o^\alpha\cdot\mathbf{F}_*^{-1} \tag{3.34}$$

The velocity gradient in the intermediate configuration is a consequence of the slip rates $\dot{\gamma}^\alpha$ over n active slip systems, such that

$$\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1}=\sum_{\alpha=1}^{n}\dot{\gamma}^\alpha\,\mathbf{s}_o^\alpha\otimes\mathbf{m}_o^\alpha \tag{3.35}$$

Using (3.34), the corresponding tensor in the deformed configuration is

$$\mathbf{F}_*\cdot(\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1})\cdot\mathbf{F}_*^{-1}=\sum_{\alpha=1}^{n}(\mathbf{P}^\alpha+\mathbf{Q}^\alpha)\,\dot{\gamma}^\alpha \tag{3.36}$$

where the second-order tensors $\mathbf{P}^\alpha$ and $\mathbf{Q}^\alpha$ are defined by

$$\mathbf{P}^\alpha=(\mathbf{s}^\alpha\otimes\mathbf{m}^\alpha)_s,\quad \mathbf{Q}^\alpha=(\mathbf{s}^\alpha\otimes\mathbf{m}^\alpha)_a \tag{3.37}$$

By decomposing the lattice velocity gradient $\mathbf{L}_*$ into its symmetric and anti-symmetric part, the lattice rate of deformation $\mathbf{D}_*$ and the lattice spin $\mathbf{W}_*$, there follows

$$\mathbf{D}=\mathbf{D}_*+\sum_{\alpha=1}^{n}\mathbf{P}^\alpha\,\dot{\gamma}^\alpha,\quad \mathbf{W}=\mathbf{W}_*+\sum_{\alpha=1}^{n}\mathbf{Q}^\alpha\,\dot{\gamma}^\alpha \tag{3.38}$$

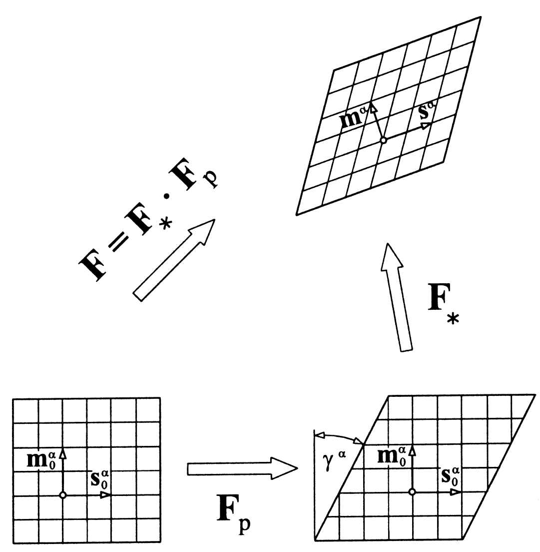

Fig. 3 Kinematic model of elastoplastic deformation of a single crystal. The material flows through the crystalline lattice by crystallographic slip, which gives rise to deformation gradient $\mathbf{F}_p$. Subsequently, the material with embedded lattice deforms elastically from the intermediate to current configuration. The corresponding deformation gradient is $\mathbf{F}_*$.

Since crystallographic slip is an isochoric deformation process, the elastic strain energy per unit initial volume can be written as $\Psi_{\mathrm{e}} = \Psi_{\mathrm{e}}(\mathbf{E}_*)$. The scalar function Ψ_{e} depends on the strain components expressed in the coordinate system with fixed orientation relative to the lattice orientation in $\mathcal{B}_{\mathrm{o}}$ and $\mathcal{B}_{\mathrm{p}}$. This is noted because for anisotropic crystals Ψ_{e} is not an isotropic scalar function of $\mathbf{E}_*$, and its representation depends on the selected coordinate system. It is also assumed that elastic properties of the crystal are not affected by the crystallographic slip. The symmetric Piola–Kirchhoff stress with respect to lattice deformation is then

$$\mathbf{T}_* = \frac{\partial \Psi_{\mathrm{e}}}{\partial \mathbf{E}_*} \tag{3.39}$$

The stress tensor $\mathbf{T}_*$ is related to the Kirchhoff stress $\boldsymbol{\tau}$ by

$$\mathbf{T}_* = \mathbf{F}_*^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}_*^{-T} \tag{3.40}$$

The plastic incompressibility is assumed, so that $\det \mathbf{F}_* = \det \mathbf{F}$. The rate of the Piola–Kirchhoff stress $\dot{\mathbf{T}}_*$ can be expressed in terms of the convected rate of the Kirchhoff stress $\overset{\blacktriangle}{\boldsymbol{\tau}}$ as [92]

$$\dot{\mathbf{T}}_* = \mathbf{F}_*^{-1} \cdot \overset{\blacktriangle}{\boldsymbol{\tau}} \cdot \mathbf{F}_*^{-T}, \quad \overset{\blacktriangle}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L}_* \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}_*^T \tag{3.41}$$

It can be readily verified that

$$\overset{\blacktriangle}{\boldsymbol{\tau}} = \overset{\triangle}{\boldsymbol{\tau}} + (\mathbf{L} - \mathbf{L}^*) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot (\mathbf{L} - \mathbf{L}^*)^T \tag{3.42}$$

where $\overset{\triangle}{\boldsymbol{\tau}}$ is the convected rate of the Kirchhoff stress with respect to total velocity gradient $\mathbf{L}$. Similarly,

$$\dot{\boldsymbol{\tau}} = \mathring{\boldsymbol{\tau}} + \sum_{\alpha=1}^{n} (\mathbf{Q}^\alpha \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{Q}^\alpha)\, \dot{\gamma}^\alpha \tag{3.43}$$

where $\dot{\boldsymbol{\tau}}$ and $\mathring{\boldsymbol{\tau}}$ are the Jaumann rates of the Kirchhoff stress with respect to the lattice and total spin ($\mathbf{W}^*$ and $\mathbf{W}$), respectively.

On the other hand, taking the time derivative in Eq. (3.39), there follows

$$\dot{\mathbf{T}}_* = \bar{\boldsymbol{\Lambda}}_* : \dot{\mathbf{E}}_*, \quad \bar{\boldsymbol{\Lambda}}_* = \frac{\partial^2 \Psi_{\mathrm{e}}}{\partial \mathbf{E}_* \otimes \partial \mathbf{E}_*} \tag{3.44}$$

Substituting the first of (3.41) into Eq. (3.44), we deduce

$$\overset{\blacktriangle}{\boldsymbol{\tau}} = \bar{\mathcal{L}}_* : \mathbf{D}_*, \quad \bar{\mathcal{L}}_* = \mathbf{F}_* \mathbf{F}_* \bar{\boldsymbol{\Lambda}}_* \mathbf{F}_*^T \mathbf{F}_*^T \tag{3.45}$$

If the Jaumann rate corotational with the lattice spin is used, Eq. (3.45) can be recast in the form

$$\dot{\boldsymbol{\tau}} = \mathcal{L}_* : \mathbf{D}_* \tag{3.46}$$

The relationship between the corresponding elastic moduli tensors is specified by an equation such as (3.10). Along elastic branch of the response (elastic unloading from an elastoplastic state), the total and the lattice velocity gradients coincide, so that $\mathbf{L}_* = \mathbf{L}$ and $\dot{\boldsymbol{\tau}} = \mathring{\boldsymbol{\tau}}$.

The rate-type constitutive framework for the elastoplastic loading of a single crystal is obtained by substituting Eq. (3.43) into Eq. (3.46). The result is

$$\mathring{\boldsymbol{\tau}} = \mathcal{L}_* : \mathbf{D} - \sum_{\alpha=1}^{n} \mathbf{C}^\alpha\, \dot{\gamma}^\alpha \tag{3.47}$$

where

$$\mathbf{C}^\alpha = \mathcal{L}_* : \mathbf{P}^\alpha + (\mathbf{Q}^\alpha \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{Q}^\alpha) \tag{3.48}$$

The elastic part of the stress rate $\mathring{\boldsymbol{\tau}}$ is

$$(\mathring{\boldsymbol{\tau}})_{\mathrm{e}} = \mathcal{L}_* : \mathbf{D} \tag{3.49}$$

since only the remaining part of the stress rate depends on the slip rates $\dot{\gamma}^\alpha$. This is the plastic part

$$(\mathring{\boldsymbol{\tau}})_{\mathrm{p}} = -\sum_{\alpha=1}^{n} \mathbf{C}^\alpha\, \dot{\gamma}^\alpha \tag{3.50}$$

For the rate-independent elastoplastic crystal, it is commonly assumed that plastic flow occurs on a slip system when the resolved shear stress $\tau^\alpha = \mathbf{P}^\alpha : \boldsymbol{\tau} = \mathbf{s} \cdot \boldsymbol{\tau} \cdot \mathbf{m}$ on that system reaches the critical value ($\tau^\alpha = \tau_{\mathrm{cr}}^\alpha$). The rate of change of the critical value of the resolved shear stress on a given slip system is specified by the hardening law

$$\dot{\tau}_{\mathrm{cr}}^\alpha = \sum_{\beta=1}^{n_0} h_{\alpha\beta}\, \dot{\gamma}^\beta, \quad \alpha = 1,2,\ldots,N \tag{3.51}$$

The total number of all available slip systems is N, while n_0 is the number of critical (potentially active) slip systems, for which $\tau^\alpha = \tau_{\mathrm{cr}}^\alpha$. The coefficients $h_{\alpha\beta}$ are the slip-plane hardening rates (moduli). The moduli corresponding to $\alpha = \beta$ represent the self-hardening on a given slip system, while $\alpha \neq \beta$ moduli represent the latent hardening. Different latent hardening theories, with the reference to original work, are examined in the book by Havner [10]. It can be shown that

$$\dot{\gamma}^\alpha = \sum_{\beta=1}^{n} g_{\alpha\beta}^{-1} \mathbf{C}^\beta : \mathbf{D} \tag{3.52}$$

where $n \leqslant n_0$ is the number of active slip systems, and

$$g_{\alpha\beta} = h_{\alpha\beta} + \mathbf{C}^\alpha : \mathbf{P}^\beta \tag{3.53}$$

In Eq. (3.52), it is assumed that the inverse matrix, whose components are designated by $g_{\alpha\beta}^{-1}$, exists. The substitution into Eq. (3.50), in conjunction with Eq. (3.49), yields the final constitutive structure for elastoplastic deformation of a single crystal

$$\mathring{\boldsymbol{\tau}} = \left(\mathcal{L}_* - \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} g_{\alpha\beta}^{-1}\, \mathbf{C}^\alpha \otimes \mathbf{C}^\beta \right) : \mathbf{D} \tag{3.54}$$

4 BIOMECHANICS

The analysis of the stress-modulated growth of living tissues and other biomaterials has been an important research topic in biomechanics during past several decades. Early work includes a study of the relationship between mechanical loads and uniform growth by Hsu [93], and a study of the mass deposition and resorption processes in hard tissues by Cowin and Hegedus [94]. The latter work provided the governing equations of the so-called adaptive elasticity theory, in which an elastic material adopts its structure to applied loading. Fundamental contribution was further made by Skalak *et al* [95] in the analytical description of the volumetrically distributed mass growth, and the mass growth by deposition or resorption on the surface. The origin and the role of the residual stresses in biological tissues have been examined both analytically and experimentally by many researchers. The review papers by Taber [96] and Humphrey [97] contain an extensive list of pertinent references. In contrast to hard tissues (bones), which undergo only small deformations, soft tissues such as blood vessels and tendons can experience large deformations. An important step toward the general analysis of finite volumetric growth of pseudo-elastic soft tissues was made by Rodrigez *et al* [11], who decomposed the total deformation gradient into its elastic and growth part. The subsequent work includes the contributions by Taber and Eggers [12], Taber and Perucchio [52], Chen and Hoger [13], Klisch and Van Dyke [14], and Lubarda and Hoger [15].

We assume that material points are everywhere dense during the volumetric mass growth, so that in any small neighborhood around the particle there are always points that existed before the growth. This assumption enables us to treat the problems of volumetric mass growth by using the usual continuum mechanics concepts, such as deformation gradient and strain tensors. The deformation gradient in the biomechanic theory of volumetric mass growth is due to both, the mass growth and the deformation caused by externally applied and the growth-induced stresses. The intermediate configuration $\mathcal{B}_g$ is defined by an instantaneous elastic destressing of the current material configuration $\mathcal{B}$ to zero stress (Fig. 4), such that

$$\mathbf{F}=\mathbf{F}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{g}} \tag{4.1}$$

This decomposition is formally analogous to the previously considered thermoelastic and elastoplastic decompositions. The modification of the decomposition to account for the residually stressed reference configuration was suggested by Hoger *et al* [53].

If the mass of an infinitesimal volume element in the initial configuration is $\mathrm{d}m^{\mathrm{o}}=\varrho^{\mathrm{o}}\,\mathrm{d}V^{\mathrm{o}}$, then the mass of the corresponding element in configurations $\mathcal{B}_g$ and $\mathcal{B}$ is

$$\mathrm{d}m=\varrho_{\mathrm{g}}\,\mathrm{d}V_{\mathrm{g}}=\varrho\,\mathrm{d}V \tag{4.2}$$

Since

$$\mathrm{d}m=\mathrm{d}m^{\mathrm{o}}+\int_0^t r_{\mathrm{g}}^{\mathrm{o}}\,\mathrm{d}\tau\,\mathrm{d}V^{\mathrm{o}} \tag{4.3}$$

where r_{g}^0 is the time rate of the mass growth per unit initial volume, and having regard to

$$\mathrm{d}V_{\mathrm{g}}=J_{\mathrm{g}}\,\mathrm{d}V^{\mathrm{o}},\quad J_{\mathrm{g}}=\det\mathbf{F}_{\mathrm{g}} \tag{4.4}$$

it follows that

$$\varrho_{\mathrm{g}}J_{\mathrm{g}}=\varrho^{\mathrm{o}}+\int_0^t r_{\mathrm{g}}^{\mathrm{o}}\,\mathrm{d}\tau \tag{4.5}$$

In addition, we have $\varrho_{\mathrm{g}}J_{\mathrm{g}}=\varrho J$ and $\varrho_{\mathrm{g}}=\varrho\,J_{\mathrm{e}}$, because $\mathrm{d}V=J_{\mathrm{e}}\,\mathrm{d}V_{\mathrm{g}}$ and $J=J_{\mathrm{e}}J_{\mathrm{g}}$. The Jacobian of the elastic deformation is $J_{\mathrm{e}}=\det\mathbf{F}_{\mathrm{e}}$.

Consider an isothermal deformation and growth process. Denote the set of structural tensors that describe the state of elastic anisotropy in both initial and intermediate configuration by $\mathbf{S}^{\mathrm{o}}$. For simplicity, it will be assumed that the state of elastic anisotropy remains unaltered during the growth and deformation processes. The elastic strain energy per unit current mass is then an isotropic function of the elastic strain $\mathbf{E}_{\mathrm{e}}$ and the tensors $\mathbf{S}^{\mathrm{o}}$, so that $\psi_{\mathrm{e}}=\psi_{\mathrm{e}}(\mathbf{E}_{\mathrm{e}},\mathbf{S}^{\mathrm{o}},\varrho_{\mathrm{g}}^{\mathrm{o}})$ and

$$\boldsymbol{\tau}=\mathbf{F}_{\mathrm{e}}\cdot\frac{\partial(\varrho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial\mathbf{E}_{\mathrm{e}}}\cdot\mathbf{F}_{\mathrm{e}}^{T}=2\mathbf{F}_{\mathrm{e}}\cdot\frac{\partial(\varrho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial\mathbf{C}_{\mathrm{e}}}\cdot\mathbf{F}_{\mathrm{e}}^{T} \tag{4.6}$$

For example, suppose that the material in the initial configuration $\mathcal{B}^0$ is characterized by an orthogonal network of fibers as orthotropic. Let the unit vectors $\mathbf{m}^0$, $\mathbf{n}^0$, and $\mathbf{m}^0\times\mathbf{n}^0$ specify the principal axes of orthotropy in both the initial and the intermediate configuration. The intermediate configuration has the same fiber orientation relative to the fixed frame of reference as does the initial configuration. The orthotropic symmetry will remain preserved during the mass growth if the fibers are embedded in the material, and if $\mathbf{F}_{\mathrm{g}}$ is defined such that $\mathbf{m}^0$ and $\mathbf{n}^0$ are its eigendirections, ie,

$$\begin{aligned}&\mathbf{F}_{\mathrm{g}}\cdot\mathbf{m}^0=\eta_{\mathrm{g}}\,\mathbf{m}^0,\quad \mathbf{F}_{\mathrm{g}}\cdot\mathbf{n}^0=\zeta_{\mathrm{g}}\,\mathbf{n}^0\\&\mathbf{F}_{\mathrm{g}}\cdot(\mathbf{m}^0\times\mathbf{n}^0)=\vartheta_{\mathrm{g}}(\mathbf{m}^0\times\mathbf{n}^0)\end{aligned} \tag{4.7}$$

The stretch ratios η_{g} and ζ_{g} are the stretch ratios in the directions $\mathbf{m}^0$ and $\mathbf{n}^0$, while ϑ_{g} is the stretch ratio in the direction $\mathbf{m}^0\times\mathbf{n}^0$. The infinitesimal fiber segments in the configuration $\mathcal{B}$ are obtained from those in the intermediate configuration by elastic embedding. For example, $\mathbf{m}=\mathbf{F}_{\mathrm{e}}$

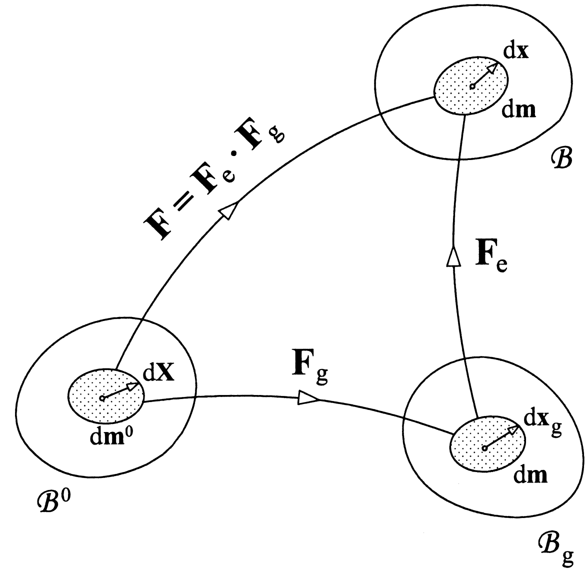

Fig. 4 Schematic representation of the multiplicative decomposition of deformation gradient into its elastic and growth parts. The mass of an infinitesimal volume element in the initial configuration $\mathcal{B}^{\mathbf{o}}$ is $\mathrm{d}m^{\mathbf{o}}$. The corresponding mass in the configurations $\mathcal{B}_{\mathrm{g}}$ and $\mathcal{B}$ is $\mathrm{d}m$.

$\cdot \mathbf{m}^0$ and $\mathbf{n}=\mathbf{F}_e \cdot \mathbf{n}^0$. The elastic strain energy per unit initial volume is in this case an isotropic function of the elastic strain tensor $\mathbf{E}_e$, and the structural tensors $\mathbf{m}^0 \otimes \mathbf{m}^0$ and $\mathbf{n}^0 \otimes \mathbf{n}^0$.

4.1 Partition of the rate of deformation

The stress-modulated growth of pseudo-elastic soft tissues is a path-dependent process, since the whole stress history during the growth process may affect the current state of the grown tissue. Thus, similarly to path-dependent elastoplasticity, we proceed with the introduction of the rate-type kinematic quantities. In view of the decomposition (4.1), the velocity gradient in the current configuration can be expressed as

$$\mathbf{L}=\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}+\mathbf{F}_e \cdot (\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) \cdot \mathbf{F}_e^{-1} \tag{4.8}$$

The symmetric and antisymmetric part of the second term on the right-hand side will be denoted by

$$\mathbf{d}_g=[\mathbf{F}_e \cdot (\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) \cdot \mathbf{F}_e^{-1}]_s, \quad \boldsymbol{\omega}_g=[\mathbf{F}_e \cdot (\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) \cdot \mathbf{F}_e^{-1}]_a \tag{4.9}$$

The total rate of deformation tensor can be additively decomposed into its elastic and plastic part, such that

$$\mathbf{D}=\mathbf{D}_e+\mathbf{D}_g \tag{4.10}$$

The elastic part of the rate of deformation tensor will be defined by the kinetic relation

$$\mathbf{D}_e=\boldsymbol{\mathcal{L}}_e^{-1}:\overset{\circ}{\boldsymbol{\tau}}, \quad \overset{\circ}{\boldsymbol{\tau}}=\dot{\boldsymbol{\tau}}-\mathbf{W}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{W} \tag{4.11}$$

where $\overset{\circ}{\boldsymbol{\tau}}$ is the Jaumann rate of the Kirchhoff stress, and $\boldsymbol{\mathcal{L}}_e$ is the instantaneous elastic moduli tensor of a considered tissue. The remaining part of the total rate of deformation ($\mathbf{D}_g=\mathbf{D}-\mathbf{D}_e$) will be referred to as the growth part of the rate of deformation. To derive an expression for $\mathbf{D}_g$, we differentiate Eq. (4.6) to obtain

$$\dot{\boldsymbol{\tau}}=(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1})\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1})^T+\mathbf{F}_e\cdot(\boldsymbol{\Lambda}_e:\dot{\mathbf{E}}_e)\cdot\mathbf{F}_e^T + \frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o} r_g^o \tag{4.12}$$

where

$$\boldsymbol{\Lambda}_e=\frac{\partial^2(\varrho_g^o\psi_e)}{\partial \mathbf{E}_e \otimes \partial \mathbf{E}_e}=4\,\frac{\partial^2(\varrho_g^o\psi_e)}{\partial \mathbf{C}_e \otimes \partial \mathbf{C}_e} \tag{4.13}$$

and

$$\frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o}=\mathbf{F}_e\cdot\frac{\partial^2(\varrho_g^o\psi_e)}{\partial \mathbf{E}_e\,\partial \varrho_g^o}\cdot\mathbf{F}_e^T=2\,\mathbf{F}_e\cdot\frac{\partial^2(\varrho_g^o\psi_e)}{\partial \mathbf{C}_e\,\partial \varrho_g^o}\cdot\mathbf{F}_e^T \tag{4.14}$$

The structural tensors $\mathbf{S}^o$ remain unchanged during the differentiation. Equivalently, Eq. (4.12) can be written as

$$\overset{\circ}{\boldsymbol{\tau}}=\boldsymbol{\mathcal{L}}_e:(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1})_s-\boldsymbol{\omega}_g\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\boldsymbol{\omega}_g+\frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o} r_g^o \tag{4.15}$$

The rectangular components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_e$ are defined by Eq. (3.10). Since

$$(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1})_s=\mathbf{D}-\mathbf{d}_g \tag{4.16}$$

Eq. (4.15) gives

$$\mathbf{D}_e=\mathbf{D}-\mathbf{d}_g-\boldsymbol{\mathcal{L}}_e^{-1}:\left(\boldsymbol{\omega}_g\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_g-\frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o} r_g^o\right) \tag{4.17}$$

According to Eq. (4.11), this is the elastic part of the rate of deformation tensor. The growth part of the rate of deformation is accordingly

$$\mathbf{D}_g=\mathbf{d}_g+\boldsymbol{\mathcal{L}}_e^{-1}:\left(\boldsymbol{\omega}_g\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\boldsymbol{\omega}_g-\frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o} r_g^o\right) \tag{4.18}$$

4.2 Isotropic mass growth

For isotropic materials, which remain isotropic during the mass growth and deformation, the elastic strain energy is an isotropic function of elastic deformation tensor $\mathbf{C}_e=\mathbf{F}_e^T\cdot\mathbf{F}_e$, ie,

$$\psi_e=\psi_e(\mathbf{C}_e,\varrho_g^o)=\psi_e(I_C,II_C,III_C,\varrho_g^o) \tag{4.19}$$

The principal invariants of $\mathbf{C}_e$ are

$$I_C=\mathrm{tr}\mathbf{C}_e, \quad II_C=\frac{1}{2}[\mathrm{tr}(\mathbf{C}_e^2)-(\mathrm{tr}\,\mathbf{C}_e)^2], \quad III_C=\det\mathbf{C}_e \tag{4.20}$$

The corresponding Kirchhoff stress follows from Eq. (4.6),

$$\boldsymbol{\tau}=2(c_2\,\mathbf{I}+c_0\,\mathbf{B}_e+c_1\,\mathbf{B}_e^2) \tag{4.21}$$

The left Cauchy–Green deformation tensor due to elastic deformation is $\mathbf{B}_e=\mathbf{F}_e\cdot\mathbf{F}_e^T$. The scalar coefficients appearing in Eq. (4.21) are

$$c_0=\frac{\partial(\varrho_g^o\psi_e)}{\partial I_C}-I_C\frac{\partial(\varrho_g^o\psi_e)}{\partial II_C}$$

$$c_1=\frac{\partial(\varrho_g^o\psi_e)}{\partial II_C}, \quad c_2=III_C\frac{\partial(\varrho_g^o\psi_e)}{\partial III_C} \tag{4.22}$$

If the mass growth takes place isotropically, the growth part of deformation gradient is

$$\mathbf{F}_g=\vartheta_g\,\mathbf{I} \tag{4.23}$$

The isotropic stretch ratio due to volumetric mass growth is denoted by ϑ_g. This is the ratio of the corresponding infinitesimal material lengths in the configurations $\mathcal{B}^o$ and $\mathcal{B}_g$. It readily follows that the velocity gradient in the intermediate configuration is

$$\dot{\mathbf{F}}_g\cdot\mathbf{F}_g^{-1}=\frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} \tag{4.24}$$

The velocity gradient in the current configuration is consequently

$$\mathbf{L}=\dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1}+\frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} \tag{4.25}$$

Since the spin tensor $\boldsymbol{\omega}_g=\mathbf{0}$ in the case of isotropic mass growth, the growth part of the rate of deformation tensor becomes

$$\mathbf{D}_g = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} - \boldsymbol{\mathcal{L}}_e^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \varrho_g^o} r_g^o \right) \tag{4.26}$$

which follows from Eq. (4.18). The explicit representation for the rectangular components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_e$ can be found in Lubarda and Hoger [15]. In contrast to phenomenological elastoplasticity, where appropriate plasticity postulates can be used to guide the construction of the constitutive expression for the plastic rate of deformation, the growth part of deformation tensor in the considered model of tissue growth is deduced from the representation of the growth part of deformation gradient, and the evolution equation for the corresponding growth stretch ratio.

Various forms of the strain energy function were proposed in the literature for different biological materials. The articles by Holzapfel *et al* [98] and Sacks [99] contain a number of pertinent references. Following Fung's [100,101] proposal for vascular soft tissues, modeled as incompressible elastic materials, the elastic strain energy per unit initial volume can be taken as

$$\varrho_g^o \psi_e = \frac{1}{2}\alpha_0[\exp(Q) - Q - 1] + \frac{1}{2}q - \frac{1}{2}p(III_C - 1) \tag{4.27}$$

Here, Q and q are the polynomials in the invariants of $\mathbf{C}_e$, which include terms up to the fourth order in elastic stretch ratios, ie,

$$Q = \alpha_1(I_C - 3) + \alpha_2(II_C - 3) + \alpha_3(I_C - 3)^2 \tag{4.28}$$

$$q = \beta_1(I_C - 3) + \beta_2(II_C - 3) + \beta_3(I_C - 3)^2 \tag{4.29}$$

The incompressibility constraint is $III_C - 1 = 0$, and the pressure p plays the role of the Lagrangian multiplier. The α's and β's are the material parameters. In order that the intermediate configuration is unstressed, it is required that $\beta_1 - 2\beta_2 = Jp$. The effects of supra-physiologic temperatures on the mechanical response of tissues is discussed by Humphrey [102].

4.3 Evolution equation for stretch ratio

The constitutive formulation is completed by specifying an appropriate evolution equation for the stretch ratio ϑ_g. In the particular, but for the tissue mechanics important special case, when the growth takes place in a density preserving manner ($\varrho_g = \varrho^o$), we have from Eq. (4.24)

$$\mathrm{tr}(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) = 3\frac{\dot{\vartheta}_g}{\vartheta_g} = \frac{r_g}{\varrho} \tag{4.30}$$

Thus, recalling that $r_g/\varrho = r_g^o/\varrho_g^o$, the rate of mass growth $r_g^o = \mathrm{d}\varrho_g^o/\mathrm{d}t$ can be expressed in terms of the rate of stretch ratio as

$$r_g^o = 3\,\varrho_g^o \frac{\dot{\vartheta}_g}{\vartheta_g} \tag{4.31}$$

Upon integration of Eq. (4.31), taking into account the initial conditions $\vartheta_g^o = 1$ and $\varrho_g^o = \varrho^o$, there follows

$$\varrho_g^o = \varrho^o \vartheta_g^3 \tag{4.32}$$

Lubarda and Hoger [15] studied the structure of the evolution equations for the stretch ratios in different types of anisotropic biomaterials. For an isotropic tissue, they proposed the following expression

$$\dot{\vartheta}_g = f_\vartheta(\vartheta_g, \mathrm{tr}\,\mathbf{T}_e) \tag{4.33}$$

The tensor $\mathbf{T}_e$ is the symmetric Piola–Kirchhoff stress with respect to intermediate configuration $\mathcal{B}_g$. Equation (4.33) in effect specifies the volume increase by mass growth, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}V_g}{\mathrm{d}V^o}\right) = (\det \mathbf{F}_g)\mathrm{tr}(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) = 3\,\vartheta_g^2\,\dot{\vartheta}_g \tag{4.34}$$

For isotropic mass growth, only the spherical part of the stress tensor $\mathbf{T}_e$ is assumed to affect the change of the stretch ratio. The spherical part of $\mathbf{T}_e$ can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ and the elastic deformation as

$$\mathrm{tr}\mathbf{T}_e = J_e\,\mathbf{B}_e^{-1} : \boldsymbol{\sigma} \tag{4.35}$$

The simplest evolution of the stretch ratio incorporates a linear dependence on stress, such that

$$\dot{\vartheta}_g = k_\vartheta(\vartheta_g)\mathrm{tr}\,\mathbf{T}_e \tag{4.36}$$

This implies that the growth-equilibrium stress is equal to zero ($\dot{\vartheta}_g = 0$ when $\mathrm{tr}\mathbf{T}_e = 0$). The coefficient k_ϑ may be constant, or dependent on ϑ_g. For example, k_ϑ may take one value during the development of the tissue, and another value during the normal maturity. Yet another value may be characteristic for abnormal conditions, such as occur in thickening of blood vessels under hypertension. To prevent an unlimited growth at non-zero stress, the following expression for the function k_ϑ in Eq. (4.36) is suggested

$$k_\vartheta(\vartheta_g) = k_{\vartheta 0}^+ \left(\frac{\vartheta_g^+ - \vartheta_g}{\vartheta_g^+ - 1} \right)^{m_\vartheta^+}, \quad \mathrm{tr}\,\mathbf{T}_e > 0 \tag{4.37}$$

where $\vartheta_g^+ > 1$ is the limiting value of the stretch ratio that can be reached by mass growth, and $k_{\vartheta 0}^+$ and m_ϑ^+ are the appropriate constants (material parameters). If the mass growth is homogeneous throughout the body, ϑ_g^+ is constant, but for a non-uniform mass growth caused by non-uniform biochemical properties, ϑ_g^+ may be different at different points (for example, inner and outer layers of an aorta may have different growth potentials, in addition to stress-modulated growth effects). It is assumed that the stress-modulated growth occurs under tension, while resorption takes place under compression. In the latter case

$$k_\vartheta(\vartheta_g) = k_{\vartheta 0}^- \left(\frac{\vartheta_g - \vartheta_g^-}{1 - \vartheta_g^-} \right)^{m_\vartheta^-}, \quad \mathrm{tr}\mathbf{T}_e < 0 \tag{4.38}$$

where $\vartheta_g^- < 1$ is the limiting value of the stretch ratio that can be reached by mass resorption. For generality, the resorption parameters $k_{\vartheta 0}^-$ and m_ϑ^- are taken to be different than those in growth.

Other evolution equations were also suggested in the literature, motivated by the possibilities of growth and resorption. The most well-known is the evolution equation for mass growth in terms of a nonlinear function of stress, which includes three growth-equilibrium states of stress [103]. The material parameters in these expressions are specified in accordance with experimental data for a particular tissue. Appealing tests include those with a transmural radial cut through the blood vessel, which relieves the residual stresses due to differential growth of its inner and outer layers. The opening angle provides a convenient measure of the circumferential residual strain, as discussed by Liu and Fung [104,105], Humphrey [97], Taber and Eggers [12], and others.

5 CONCLUSIONS

Some fundamental issues in the formulation of constitutive theories of material response based on the multiplicative decomposition of the deformation gradient are reviewed in this paper. Large deformations of thermoelastic and elastoplastic materials are considered, as well as large growth-induced deformations of pseudo-elastic soft tissues. The use of the multiplicative decomposition of the deformation gradient in thermoelasticity and phenomenological polycrystalline plasticity can be regarded to large extent as optional, since these constitutive formulations can also proceed without the introduction of the decomposition (eg, Truesdell and Noll [106] for thermoelasticity and Hill [71] for elastoplasticity). Some of the results derived on the basis of thermoelastic decomposition, however, appear to be more suitable for the incorporation of experimental data for the temperature dependent elastic moduli, thermal expansion, and specific heats (Section 2). The kinematic and kinetic aspects of the partition of the stress and strain rates in phenomenological elastoplasticity are richer or more illuminating when addressed in the framework of the multiplicative decomposition, which was discussed in Section 3. This is particularly the case when large elastic deformations accompany plastic deformations, as occurs under high pressure dynamic loading. Furthermore, there is an important application of the multiplicative decomposition of the deformation gradient in damage-elastoplasticity [66,67,107], where plastic deformation significantly affects the initial elastic properties of the material. The multiplicative decomposition was also successfully employed in the constitutive analysis of various polymeric materials [108–112]. In monocrystalline plasticity, the multiplicative decomposition of the deformation gradient is regarded and commonly adopted as the most suitable framework to cast the constitutive analysis of large slip-induced elastoplastic deformation of single crystals [8–10]. The application of the multiplicative decomposition to the study of the stress-modulated growth of pseudo-elastic soft tissues, such as blood vessels and tendons, is more recent and least explored. This was reviewed in Section 4. The extent of the utility of the decomposition for such problems and its possible advantages, in spite of some early promising results by Klisch and Van Dyke [14] and Lubarda and Hoger [15], remain to be seen.

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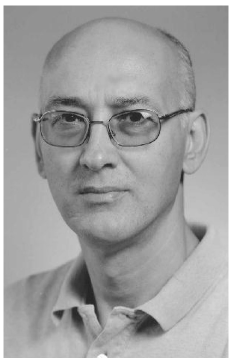

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